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Green formulas in anticipating stochastic calculus

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Abstract

In this paper we state Green type formulas for nonadapted processes with respect to "anticipating semimartingales", say U, in the Stratonovich and the Skorohod formulation. To this end we develop some notions of anticipating stochastic calculus with respect to U, based on *multiple* and *line* integrals. The basic ingredients are a *Hu–Meyer formula* for multiple integrals with respect to U and a *Fubini theorem* for the multiple Stratonovich integral.

Keywords: Anticipating stochastic calculus; Skorohod integral; Stratonovich integral; Trace; Green formulas

0. Introduction

This paper deals with problems concerning anticipating stochastic calculus with respect to generalized (nonadapted) semimartingales. Let $T = [0,1]^2$ and $W = \{W_z, z \in T\}$ be the Brownian sheet. Consider stochastic processes $u = \{u_z, z \in T\}$, $v = \{v_z, z \in T\}$, not necessarily adapted to the natural filtration associated with W. Set

$$U_z = \int_{R_z} u_r \circ \mathrm{d}W_r + \int_{R_z} v_r \,\mathrm{d}r, \tag{0.1}$$

$$V_z = \int_{R_z} u_r \,\mathrm{d}W_r + \int_{R_z} v_r \,\mathrm{d}r, \tag{0.2}$$

where $R_z = (0, z]$, $z \in T$, where the stochastic integral in (0.1) (resp. (0.2)) is a Stratonovich (resp. Skorohod) integral.

Our aim has been to establish a Green–Stratonovich formula for anticipating processes with respect to U as well as its Skorohod counterpart (see Theorem 2.3.4). That means, a formula relating "line" and "surface" stochastic integrals with respect to U (resp. V). The case of adapted integrands and $v \equiv 0$, $u \equiv 1$ has been studied in the basic paper on two-parameter processes by Cairoli and Walsh (1975). The same

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situation, but allowing nonadaptedness of the integrands, has been recently considered in Solé and Utzet (1991). The extension presented in this paper needs as first ingredients the notions of multiple integrals for anticipating processes, in the Stratonovich and the Skorohod sense, with respect to processes given by (0.1) or (0.2), as well as line integrals.

Fubini's theorem seems to be a useful tool for proving Green's formula. Although that result is almost obvious for the multiple Skorohod integral, it is not the case for the Stratonovich one. A way to establish the iteration property for the Stratonovich integral consists in analyzing first the relation between the multiple Skorohod and Stratonovich integrals and then trying to transfer the iteration property from Skorohod to Stratonovich. This program entails essentially two problems: the study of trace type terms and the iteration of such traces.

Here is a brief description of the sections of this work. Section 1 is devoted to introducing the different kinds of anticipating stochastic integrals we have mentioned before; we also define and study the "trace" terms relating the Skorohod and Stratonovich formulation. In particular, we prove an iteration property for these terms. Section 2 is devoted to proving the result which motivates this work: a Green–Stratonovich formula. First we give a precise formula relating the second-order Skorohod and Stratonovich integral with respect to U. This result is analogous to a classical one by Hu and Meyer relating the multiple Itô and Stratonovich integral (see Delgado and Sanz-Solé (1992) for an extension to anticipating integrands) and, for this reason, we call it a Hu–Meyer type formula. With this tool and the iteration property of the traces we are able to prove a Fubini theorem for the Stratonovich integral with respect to U and, then, the Green formula follows easily. Finally, as an application, we derive an Itô–Stratonovich formula (see Thieullen (1991) for related work). We close the paper with some remarks and indications on the Skorohod analogue of our results.

1. Anticipating stochastic integrals for generalized semimartingales

In this section we introduce the notion of multiple anticipating integrals, in the Skorohod and in the Stratonovich sense, with respect to processes written in the form (0.2) or (0.1), respectively. In both cases we try to follow the same pattern as for integrals with respect to the Wiener process. We define an operator giving the first-order Skorohod integral with respect to V and we identify its adjoint operator, denoted by D^V. The adjoint of the kth iterate of D^V provides a definition for the kth-order Skorohod integral with respect to V. The definition of the multiple Stratonovich integral with respect to U is the natural extension of Stratonovich integrals with respect to W (see, for instance, Delgado and Sanz-Solé 1992; Nualart and Pardoux, 1988; Solé and Utzet, 1990). It is important to know for which classes of processes the Stratonovich integral can be defined and related with the Skorohod integral. A first result in this direction has been given in Nualart and Pardoux (1988, Theorem 7.3). The essential ingredient is the existence of some trace type term. We analyze this concept in the kth-order situation, in particular we present an integral representation for the traces. This result provides a useful tool which allows us to state a property on iteration of traces.

We start with some preliminaries and basic notation. Let $(T, \mathscr{T})$ be a measurable, separable space endowed with a finite, atomless measure μ and let $W = \{W(B), B \in \mathscr{T}\}$ be a Gaussian process defined on some probability space $(\Omega, \mathscr{F}, P)$, with zero mean and covariance given by $E[W(B_1)W(B_2)] = \mu(B_1 \cap B_2)$, $B_1, B_2 \in \mathscr{T}$.

Assume that $\mathscr{F}$ is the σ-field generated by the process W. For any functional $F \in L^2(\Omega, \mathscr{F}, P)$ we have the Wiener chaos expansion

$$F = \sum_{m=0}^{\infty} I_m(f_m),$$

with $I_0(f_0) = E(F)$, and where $I_m(f_m)$ denotes the multiple Itô–Wiener integral of a symmetric function $f_m \in L^2(T^m)$. In the sequel $\underline{t} = (t_1, \ldots, t_n)$ denotes a point in T^n.

For any $k \in \mathbb{N}$ the kth *Malliavin derivative* of $F \in L^2(\Omega)$ is defined as the element of $L^2(T^k \times \Omega)$ given by

$$D_{\underline{t}}^k F = \sum_{m=k}^{\infty} \frac{m!}{(m-k)!} I_{m-k}(f_{m(\underline{t}, \cdot)}),$$

whenever the series converges in $L^2(T^k \times \Omega)$. The domain of the operator D^k is

$$\mathbb{D}^{k,2} = \left\{ F \in L^2(\Omega) : \sum_{j=1}^{k} (E[\|D^j F\|_{L^2(T^j)}^2])^{1/2} < \infty \right\}, \tag{1.1}$$

and is dense in $L^2(\Omega)$. Furthermore, D^k defines a closed operator. Its adjoint operator is the kth Skorohod integral, which is denoted by δ^k.

In the sequel k and n will denote positive integers. Moreover, we will consider the case $T = [0,1]^2$ with the natural order structure defined coordinatewise. The following notation will be used. Given $z = (s,t)$ and $z' = (s',t')$, $z, z' \in [0,1]^2$, we write $z \perp z'$ if $s \leqslant s'$ and $t \geqslant t'$, and in this case, $z \vee z' = (s', t)$. We will deal with the following order relations:

$$z R^1 z' \Leftrightarrow z \leqslant z', \qquad z R^2 z' \Leftrightarrow z \perp z',$$

$$z R^3 z' \Leftrightarrow z' \leqslant z, \qquad z R^4 z' \Leftrightarrow z' \perp z.$$

For any $z \in [0,1]^2$, R_z denotes the rectangle $(0, z] = \{z' : 0 < z' \leqslant z\}$.

In most of the situations concerning the Skorohod integral it is more convenient to deal with some subspaces of $Dom\, \delta^k$. The next definitions collect those which are needed in our work.

Definition 1.1. (a) $\mathbb{L}_k^{n,2}$ is the set of processes in $L^2(T^k; \mathbb{D}^{nk,2})$.

(b) $\mathbb{L}_k^{n,\infty}$ is the set of processes $X \in L^\infty(T^k \times \Omega)$ such that $\|D^{kh} X\|_{L^\infty(T^{k(h+1)} \times \Omega)} < \infty$, for any $1 \leqslant h \leqslant n$.

The spaces $\mathbb{L}_k^{n,2}$ are used in the stochastic calculus concerning the kth Skorohod integral. By convention $\mathbb{L}_0^{n,2} = L^2(\Omega)$. For $k = 1$ we will drop the subscript k in (a) and (b).

Consider stochastic processes $u = \{u_z,\ z \in T\}$, $v = \{v_z,\ z \in T\}$ satisfying the following conditions: For any $z \in T$, the process $u\mathbf{1}_{R_z}$ belongs to $Dom\,\delta$ and $\int_T |v_z|\,dz < \infty$ a.s. Set $V = (u, v)$ to denote

$$V_z = \int_{R_z} u_r \,dW_r + \int_{R_z} v_r \,dr. \tag{1.2}$$

Our next aim is to define the multiple Skorohod integral of a process X in $L^2(T^k \times \Omega)$ with respect to the process $V = (u, v)$.

Let $X \in L^2(T \times \Omega)$ be such that $Xu \in Dom\,\delta$ and $\int_T |X_r v_r|\,dr < \infty$, a.s. We define the Skorohod integral of X with respect to V as

$$\delta^V(X) := \delta(Xu) + \int_T X_r v_r \,dr. \tag{1.3}$$

By Theorem 3.1 of Jolis and Sanz-Solé (1992), if $u \in \mathbb{L}^{1,\infty}$ and $v \in L^2(T \times \Omega)$, then $\mathbb{L}^{1,2} \subset Dom\,\delta^V$. Hence δ^V defines an operator on $L^2(T \times \Omega)$ taking values in $L^2(\Omega)$ whose domain is dense in $L^2(T \times \Omega)$. Denote by D^V the adjoint operator. Then $Dom\,D^V$ is the set of elements $F \in L^2(\Omega)$ such that there exists $c \geqslant 0$ with

$$|E(F\delta^V(X))| \leqslant c\,\|X\|_{L^2(T \times \Omega)},$$

for any $X \in Dom\,\delta^V$. Moreover, for any $F \in Dom\,D^V$, $D^V F$ is an element of $L^2(T \times \Omega)$ uniquely determined by the duality relation

$$E(F\delta^V(X)) = \langle D^V F, X\rangle_{L^2(T \times \Omega)},$$

for any $X \in Dom\,\delta^V$.

Let $X \in \mathbb{L}^{1,2}$ and $F \in Dom\,D$. Then (1.3) and the duality between δ and D yield

$$\begin{aligned} E(F\delta^V(X)) &= E\left(F\delta(Xu) + F\int_T X_r v_r \,dr\right) \\ &= E\int_T [(D_r F) X_r u_r + F v_r X_r]\,dr. \end{aligned}$$

Consequently, $F \in Dom\,D^V$ and

$$D_r^V F = u_r D_r F + F v_r. \tag{1.4}$$

We define $D^{k,V}$ by iteration, that means,

$$D^{k,V} = D^V \circ \overset{k}{\cdots} \circ D^V.$$

Using (1.4) it is easy to prove that, if for a.e. $t \in T$, the random variables u_t and v_t belong to $Dom\,D^{k-1}$, then $Dom\,D^k \subset Dom\,D^{k,V}$. Consequently, $Dom\,D^{k,V}$ is dense in $L^2(\Omega)$.

Definition 1.2. Let $V = (u, v)$ with $u \in \mathbb{L}^{1,\infty}$ and $v \in L^\infty(T \times \Omega)$. Fix $k \in \mathbb{Z}^+$. Assume that for a.e. t the random variables u_t and v_t belong to $Dom\,D^{k-1}$. We define the *multiple Skorohod integral of order k with respect to V*, δ_k^V, as the adjoint operator of $D^{k,V}$. More precisely δ_k^V is an operator defined on $L^2(T^k \times \Omega)$, taking values on $L^2(\Omega)$, whose domain is given by the elements $X \in L^2(T^k \times \Omega)$ such that there exists $c \geqslant 0$ with

$$\langle D^{k,V}F, X\rangle_{L^2(T^k \times \Omega)} \leqslant c\|F\|_{L^2(\Omega)},$$

for any $F \in Dom\, D^{k,V}$.

Furthermore, if $X \in Dom\, \delta_k^V$, $\delta_k^V(X)$ is determined by the duality relation

$$E(F\delta_k^V(X)) = \langle D^{k,V}F, X\rangle_{L^2(T^k \times \Omega)}. \tag{1.5}$$

We will also use the notation $\int_{T^k} X dV \overset{k}{\because} dV$ for $\delta_k^V(X)$.

The next proposition establishes a Fubini theorem for the Skorohod integral introduced in the preceding definition. Its proof is based on (1.5) and is left to the reader.

Proposition 1.3. *Let $X \in L^2(T^{k+j} \times \Omega)$ for some integers k, $j \geqslant 1$, $V = (u, v)$ with $u \in \mathbb{L}^{1,\infty}$ and $v \in L^\infty(T \times \Omega)$. Furthermore, assume*

(a) *for a.e. $t \in T$, u_t, $v_t \in Dom\, D^{k+j-1}$,*
(b) *for a.e. $\underline{s} \in T^k$, $X(\underline{s}, \cdot) \in Dom\, \delta_j^V$,*
(c) *$\delta_j^V(X(\underline{s}, \cdot)) \in Dom\, \delta_k^V$.*

Then $X \in Dom\, \delta_{k+j}^V$ and $\delta_{k+j}^V(X) = \delta_k^V(\delta_j^V(X))$.

Assume $u \in \mathbb{L}^{k,\infty}$ and $v \in \mathbb{L}^{k-1,\infty}$, $k \geqslant 1$, where $\mathbb{L}^{0,\infty} = L^\infty(T \times \Omega)$. The preceding proposition shows inductively that, under those hypotheses, $\mathbb{L}_k^{1,2} \subset Dom\, \delta_k^V$.

Let us now consider the Stratonovich case. First we need some notation. We denote by π any partition of $T = [0, 1]^2$, $\pi = \{z_i, i = 1, \ldots, r_\pi\}$. The rectangles determined by the points z_i of the grid will be denoted by Δ_i and by Δ_i^2 the set $\Delta_i \times \Delta_i$. For a real function f defined on T, $f(\Delta_i)$ denotes the increment of f in the sense of distribution functions. Given a Borel set A of T, $|A|$ means its Lebesgue measure. The grid π is determined by the product of two grids $\pi_{(1)} = \{0 = s_1 < \cdots < s_{r_1+1} = 1\}$, $\pi_{(2)} = \{0 = t_1 < \cdots < t_{r_2+1} = 1\}$ of $[0, 1]$. We define the mesh of π as

$$|\pi| = \left\{\max_{1 \leqslant i \leqslant r_1} |s_{i+1} - s_i| + \max_{1 \leqslant j \leqslant r_2} |t_{j+1} - t_j|\right\}.$$

We recall the definition of the simple Stratonovich integral (see Definition 7.1 in Nualart and Pardoux, 1988). A process $X = \{X_t, t \in T\}$ in $L^2(T \times \Omega)$ is said to be *Stratonovich integrable* if

$$\left\{S_\pi(X) = \sum_{i=1}^{r_\pi} \frac{1}{|\Delta_i|}\left(\int_{\Delta_i} X_t \mathrm{d}t\right) W(\Delta_i),\ \pi \text{ partition of } T\right\}$$

converges in $L^2(\Omega)$ as $|\pi| \to 0$. The set of processes Stratonovich integrable is denoted by $Dom\, I^s$. Assume now that $u\mathbf{1}_{R_z}$ belongs to $Dom\, I^s$ for any $z \in T$ and $\int_T |v_z| \mathrm{d}z < \infty$

a.s. Let $U = (u^s, v)$ be the process defined by

$$U_z = \int_{R_z} u_r \circ \mathrm{d}W_r + \int_{R_z} v_r \mathrm{d}r. \tag{1.6}$$

Definition 1.4. Let $X = \{X_{\underline{t}}, \underline{t} \in T^k\}$ be a stochastic process in $L^2(T^k \times \Omega)$ and π a partition of T. Set

$$S_\pi^{U,k}(X) = \sum_{i_1, \ldots, i_k = 1}^{r_\pi} \frac{1}{|\Delta_{i_1}| \cdots |\Delta_{i_k}|} \left(\int_{\Delta_{i_1} \times \cdots \times \Delta_{i_k}} X_{\underline{t}} \mathrm{d}\underline{t} \right) U(\Delta_{i_1}) \cdots U(\Delta_{i_k}). \tag{1.7}$$

X is said to be *k-Stratonovich integrable* with respect to U if the family $\{S_\pi^{U,k}(X)$, π partition of $T\}$ converges in $L^2(\Omega)$ as $|\pi| \to 0$. We call this limit the k-Stratonovich integral of the process X with respect to $U = (u^s, v)$, and it will be denoted by $I_k^{s,U}(X)$.

We also use the notation $\int_{T^k} X \circ \mathrm{d}U \circ \overset{k}{\ldots} \circ \mathrm{d}U$ for $I_k^{s,U}(X)$. The set of processes for which such an integral can be defined is denoted by $Dom\, I_k^{s,U}$. We will write $\mathrm{I}^{s,U}(X)$ when $k = 1$.

Remarks.

(1.5) In the preceding definition, the random variable $I_k^{s,U}(X)$ can be obtained, equivalently, as the limit of $\{S_{\pi_n}^{U,k}(X), n \geqslant 1\}$ for any increasing sequence of partitions $\{\pi_n, n \geqslant 1\}$ of T such that $\lim_{n\to\infty} |\pi_n| = 0$.

(1.6) The process X belongs to $Dom\, I_k^{s,U}$ if and only if its symmetrization $\tilde{X}$ belongs to $Dom\, I_k^{s,U}$. In this case, $I_k^{s,U}(X) = I_k^{s,U}(\tilde{X})$.

(1.7) Assume that in (1.6), $u \equiv 1$ and $v \equiv 0$. Some results concerning the multiple Stratonovich integral in this particular case are given in Solé and Utzet (1990) and Delgado and Sanz-Solé (1992). This integral is denoted by I_k^s.

The Stratonovich anticipating calculus needs more restrictive spaces than those introduced previously, in order to ensure the existence of traces in a sense to be specified later. Indeed, let us recall some known facts.

Definition 1.8. We denote by $\mathbb{L}_c^{1,2}$ the set of processes X in $\mathbb{L}^{1,2}$ such that

(1.8.1) $\operatorname{ess\,sup}_{t,s} \|D_t X_s\|_{L^2(\Omega)} < \infty$.

(1.8.2) There exists a neighborhood V of the set $\{(t,s) \in T^2: t = s\}$ such that, for any $i \in \{1,2,3,4\}$, there exists a version of DX such that the mapping $(t,s) \mapsto D_t X_s$, defined on $V \cap \{(t,s) \in T^2: s R^i t\}$ and taking its values on $L^2(\Omega)$, is continuous in the variable s uniformly in t.

For $X \in \mathbb{L}_c^{1,2}$ we can define

$$(D^{(i)}X)_t = L^2(\Omega) - \lim_{s \to t} D_t X_s, \quad s\, R^i\, t, \tag{1.8}$$

and

$$(\nabla_0^1 X)_t = \frac{1}{4} \sum_{i=1}^{4} (D^{(i)}X)_t. \tag{1.9}$$

Notice that $(\nabla_0^1 X)_t \in L^2(\Omega)$.

Proposition 4.4 and Theorem 4.7 in Solé and Utzet (1991) show that, if $X \in \mathbb{L}_c^{1,2}$, then $X \in Dom\, I^s$. Moreover the trace of X, $T_{0,1}(X)$, that is, the limit in $L^2(\Omega)$ of the sequence

$$\sum_i \frac{1}{|\Delta_i|} \int_{\Delta_i^2} D_r X_s \,\mathrm{d}r\,\mathrm{d}s$$

exists and

$$I^s(X) = \delta(X) + T_{0,1}(X).$$

Furthermore

$$T_{0,1}(X) = \int_T (\nabla_0^1 X)_r \,\mathrm{d}r.$$

For $T = [0,1]$ the analogue of the space $\mathbb{L}_c^{1,2}$ has been introduced in Nualart and Pardoux (1988, Definition 7.2).

The next proposition relates the simple Stratonovich integral with respect to $U = (u^s, v)$ of a process $X \in \mathbb{L}_c^{1,2}$ with integrals of the Skorohod type.

Proposition 1.9. *Let X be a process belonging to $\mathbb{L}_c^{1,2}$ and $U = (u^s, v)$. Assume $u \in \mathbb{L}_c^{1,2} \cap \mathbb{L}^{1,\infty}$, $D^{(i)}u, v \in L^\infty(T \times \Omega)$, for any $i = 1,2,3,4$. Then, $X \in Dom\, I^{s,U} \cap Dom\, \delta^U$ and*

$$I^{s,U}(X) = \delta(Xu) + \int_T (\nabla_0^1 X)_r u_r \,\mathrm{d}r + \int_T X_r (\nabla_0^1 u)_r \,\mathrm{d}r + \int_T X_r v_r \,\mathrm{d}r \tag{1.10}$$

$$= \delta^U(X) + \int_T (\nabla_0^1 X)_r u_r \,\mathrm{d}r. \tag{1.11}$$

If, in addition, $Xu \in \mathbb{L}_c^{1,2}$, u and X are continuous in $L^2(\Omega)$ and $\operatorname{ess\,sup}_t(\|X_t\|_{L^2(\Omega)}) < \infty$, *then*

$$I^{s,U}(X) = I^s(Xu) + \int_T X_r v_r \,\mathrm{d}r. \tag{1.12}$$

Proof. Fix a sequence of grids of T, $\{\pi_n, n \geqslant 1\}$, $\pi_n = \{z_i, i = 1, \dots, r_n\}$, whose mesh tends to zero as $n \to \infty$. We will prove the convergence in $L^2(\Omega)$ as $n \to \infty$ of the sequence

$$S_{\pi_n}^{U,1}(X) = \sum_{i=1}^{r_n} \frac{1}{|\Delta_i|}\left(\int_{\Delta_i} X_z\,\mathrm{d}z\right) U(\Delta_i), \quad n \geqslant 1,$$

with

$$U(\Delta_i) = I^s(\mathbf{1}_{\Delta_i} u) + \int_{\Delta_i} v_r\,\mathrm{d}r.$$

Proposition 4.4 and Theorem 4.7 in Solé and Utzet (1991) yield

$$U(\Delta_i) = \delta(\mathbf{1}_{\Delta_i} u) + \int_{\Delta_i} [(\nabla_0^1 u)_r + v_r]\,\mathrm{d}r.$$

Set

$$A_1^n = \sum_{i=1}^{r_n} \frac{1}{|\Delta_i|}\left(\int_{\Delta_i} X_z\,\mathrm{d}z\right)\delta(\mathbf{1}_{\Delta_i} u),$$

$$A_2^n = \sum_{i=1}^{r_n} \frac{1}{|\Delta_i|}\left(\int_{\Delta_i} X_z\,\mathrm{d}z\right)\int_{\Delta_i} (\nabla_0^1 u)_r\,\mathrm{d}r,$$

$$A_3^n = \sum_{i=1}^{r_n} \frac{1}{|\Delta_i|}\left(\int_{\Delta_i} X_z\,\mathrm{d}z\right)\int_{\Delta_i} v_r\,\mathrm{d}r.$$

Then,

$$\begin{aligned} A_1^n &= \sum_{i=1}^{r_n} \frac{1}{|\Delta_i|}\left\{\delta\left(\left(\int_{\Delta_i} X_z\,\mathrm{d}z\right)\mathbf{1}_{\Delta_i} u\right) + \int_T D_t\left(\int_{\Delta_i} X_z\,\mathrm{d}z\right)\mathbf{1}_{\Delta_i}(t) u_t\,\mathrm{d}t\right\} \\ &= \delta\left(\sum_{i=1}^{r_n} \frac{1}{|\Delta_i|}\left(\int_{\Delta_i} X_z\,\mathrm{d}z\right)\mathbf{1}_{\Delta_i} u\right) + \sum_{i=1}^{r_n} \frac{1}{|\Delta_i|}\int_{\Delta_i^2} D_t X_z u_t\,\mathrm{d}z\,\mathrm{d}t. \end{aligned}$$

Since u and Du are bounded, Proposition 3.5 in Jolis and Sanz-Solé (1990) yields

$$\sum_{i=1}^{r_n} \frac{1}{|\Delta_i|}\left(\int_{\Delta_i} X_z\,\mathrm{d}z\right)\mathbf{1}_{\Delta_i} u \to X u$$

in the convergence of $\mathbb{L}^{1,2}$ as $n \to \infty$. Consequently,

$$\delta\left(\sum_{i=1}^{r_n} \frac{1}{|\Delta_i|}\left(\int_{\Delta_i} X_z\,\mathrm{d}z\right)\mathbf{1}_{\Delta_i} u\right) \to \delta(X u)$$

as $n \to \infty$, in $L^2(\Omega)$.

Since $u \in L^\infty(T \times \Omega)$, a slight modification of the proof of Theorem 4.7 in Solé and Utzet (1991) shows

$$\sum_{i=1}^{r_n} \frac{1}{|\Delta_i|}\int_{\Delta_i^2} D_t X_z u_t\,\mathrm{d}z\,\mathrm{d}t \to \int_T (\nabla_0^1 X)_z u_z\,\mathrm{d}z$$

in $L^2(\Omega)$, as n goes to infinity. Hence

$$L^2 - \lim_{n\to\infty} A_1^n = \delta(Xu) + \int_T (\nabla_0^1 X)_z u_z \,\mathrm{d}z. \tag{1.13}$$

The term A_2^n can be written as follows:

$$A_2^n = \int_T \sum_{i=1}^{r_n} \frac{1}{|\Delta_i|}\left(\int_{\Delta_i} X_z \,\mathrm{d}z\right)(\nabla_0^1 u)_r \mathbf{1}_{\Delta_i}(r)\,\mathrm{d}r.$$

Since

$$\sum_{i=1}^{r_n} \frac{1}{|\Delta_i|}\left(\int_{\Delta_i} X_z \,\mathrm{d}z\right)(\nabla_0^1 u)\mathbf{1}_{\Delta_i} \to X(\nabla_0^1 u)$$

in $L^2(T\times\Omega)$, as n goes to infinity, it follows

$$L^2 - \lim_{n\to\infty} A_2^n = \int_T X_z (\nabla_0^1 u)_z \,\mathrm{d}z. \tag{1.14}$$

Also

$$L^2 - \lim_{n\to\infty} A_3^n = \int_T X_z v_z \,\mathrm{d}z, \tag{1.15}$$

by the same arguments as before. Then, the convergences stated in (1.13), (1.14) and (1.15) yield the existence of the limit in $L^2(\Omega)$ for the sequence $\{S_{\pi_n}^{U,1}(X), n \geqslant 1\}$ as well as its value. Hence $X \in Dom\, I^{s,U}$ and (1.10) holds. It is also clear that $X \in Dom\,\delta^U$ and $U = (u, v + \nabla_0^1 u)$. Consequently, Eq. (1.11) is satisfied.

Let us now finally prove (1.12). Proposition 4.4 and Theorem 4.7 in Solé and Utzet (1991) ensure

$$I^s(Xu) = \delta(Xu) + \int_T (\nabla_0^1(Xu))_r \,\mathrm{d}r.$$

Moreover

$$\int_T (\nabla_0^1(Xu))_r \,\mathrm{d}r = \int_T [(\nabla_0^1 X)_r u_r + X_r(\nabla_0^1 u)_r]\,\mathrm{d}r.$$

Indeed, $D_r(X_t u_t)$ converges to $(D^{(i)}X)_r u_r + X_r(D^{(i)}u)_r$ in probability as $t \to r$, $t\,R^i\,r$, for any $i \in \{1,2,3,4\}$. Consequently (1.12) follows from (1.10). □

In Proposition 2.1.5 of Section 2.1 we will prove an extension of (1.11) for double integrals. This generalization is not simple. Indeed, to any stochastic process X indexed by T^2 or, in general, by T^k, we can associate different types of "traces" obtained, roughly speaking, by contraction of indices of the process itself or of the derivatives and the process together. In the next two definitions we precise this idea and we introduce the spaces where the traces exist (the analogue of $\mathbb{L}_c^{1,2}$). We start by quoting a definition given in Delgado and Sanz-Solé (1992).

Definition 1.10. Let $X = \{X_{\underline{t}}, \underline{t} \in T^k\}$ be a symmetric process belonging to $\mathbb{L}_k^{1,2}$. Fix $j \in \{0, 1, \dots, [k/2]\}$ and $r \in \{0, 1, \dots, k-2j\}$ not simultaneously zero. The trace $T_{j,r}(X)$ is the process of $\mathbb{L}_{k-2j-r}^{1,2}$ given by the $\mathbb{L}_{k-2j-r}^{1,2}$-limit of the sequence of processes

$$\left\{ \sum_{i_1, \dots, i_{j+r}} \frac{1}{|\Delta_{i_1}| \cdots |\Delta_{i_{j+r}}|} \int_{\Delta_{i_1}^2 \times \cdots \times \Delta_{i_{j+r}}^2} D_{\underline{s}}^r X_{\underline{t}} \, dt_1 \cdots dt_{2j} \, ds_1 \, dr_{2j+1} \cdots ds_r \, dt_{2j+r} \right\}$$

corresponding to a sequence of grids $\{\pi_n, n \geq 1\}$ of $T = [0,1]^2$ whose mesh tends to zero as $n \to \infty$, whenever this limit exists.

By convention, $T_{0,0}(X) = X$.

Now we introduce the spaces where the traces exist.

Definition 1.11. Let $j \in \{0, 1, \dots, [k/2]\}$ and $r \in \{0, 1, \dots, k-2j\}$. Set $\mathbb{L}_{k,c(0,0)}^{1,2} = \mathbb{L}_k^{1,2}$ and, for j and r not simultaneously zero, we denote by $\mathbb{L}_{k,c(j,r)}^{1,2}$ the set of symmetric processes in $\mathbb{L}_k^{1,2}$ such that

(1.11.1) $$\operatorname*{ess\,sup}_{(t_1, \dots, t_{2j}, t_{2j+1}, \dots, t_{2j+r}, s_1, \dots, s_r)} \|D_{t_{2j+1} \cdots t_{2j+r}}^r X_{(t_1, \dots, t_{2j}, s_1, \dots, s_r, \cdot)}\|_{\mathbb{L}_{k-2j-r}^{1,2}} < \infty.$$

(1.11.2) There exists a neighborhood $V_{j,r}$ of the set

$$\{(t_1, \dots, t_{2j}, t_{2j+1}, \dots, t_{2j+r}, s_1, \dots, s_r) \in T^{2(j+r)}:$$
$$t_1 = t_{j+1}, \dots, t_j = t_{2j}, t_{2j+1} = s_1, \dots, t_{2j+r} = s_r\}$$

such that, for any $(i_1, \dots, i_r, h_1, \dots, h_j) \in \{1,2,3,4\}^{r+j}$, there exists a version of $D^r X$ such that the mapping

$$(t_1, \dots, t_{2j}, t_{2j+1}, \dots, t_{2j+r}, s_1, \dots, s_r) \mapsto D_{t_{2j+1} \cdots t_{2j+r}}^r X_{(t_1, \dots, t_{2j}, s_1, \dots, s_r, \cdot)},$$

defined on

$$V_{j,r} \cap \{(t_1, \dots, t_{2j}, t_{2j+1}, \dots, t_{2j+r}, s_1, \dots, s_r) \in T^{2(j+r)}: s_1 R^{i_1} t_{2j+1}, \dots,$$
$$s_r R^{i_r} t_{2j+r}, t_1 R^{h_1} t_{j+1}, \dots, t_j R^{h_j} t_{2j}\},$$

and taking its values on $\mathbb{L}_{k-2j-r}^{1,2}$, is continuous in the variables $(t_1, \dots, t_j, s_1, \dots, s_r)$ uniformly in $(t_{j+1}, \dots, t_{2j}, t_{2j+1}, \dots, t_{2j+r})$.

For $X \in \mathbb{L}_{k,c(j,r)}^{1,2}$ we can define

$$(D_{(h_1 \cdots h_j)}^{(i_1 \cdots i_r)} X)_{(t_{j+1}, \dots, t_{2j+r}, \cdot)} = \mathbb{L}_{k-2j-r}^{1,2} - \lim D_{t_{2j+1} \cdots t_{2j+r}}^r X_{(t_1, \dots, t_{2j}, s_1, \dots, s_r, \cdot)}, \tag{1.16}$$

as $(s_1, \dots, s_r) \to (t_{2j+1}, \dots, t_{2j+r})$ with $s_1 R^{i_1} t_{2j+1}, \dots, s_r R^{i_r} t_{2j+r}$, and $(t_1, \dots, t_j) \to (t_{j+1}, \dots, t_{2j})$ with $t_1 R^{h_1} t_{j+1}, \dots, t_j R^{h_j} t_{2j}$. We set

$$(\nabla_j^r X)_{(t_{j+1}, \dots, t_{2j+r}, \cdot)} = \frac{1}{4^{j+r}} \sum (D_{(h_1 \cdots h_j)}^{(i_1 \cdots i_r)} X)_{(t_{j+1}, \dots, t_{2j+r}, \cdot)}, \tag{1.17}$$

where the sum extends to $i_1, \dots, i_r, h_1, \dots, h_j \in \{1,2,3,4\}$. Notice that $(\nabla^r_j X)_{(t_{j+1}, \dots, t_{2j+r}, \cdot)} \in \mathbb{L}^{1,2}_{k-2j-r}$. Set

$$\mathbb{L}^{1,2}_{k,c} = \bigcap_{j=0}^{[k/2]} \bigcap_{r=0}^{k-2j} \mathbb{L}^{1,2}_{k,c(j,r)}.$$

For a process X in $\mathbb{L}^{1,2}_{k,c}$ all traces $T_{j,r}(X)$, $j \in \{0, 1, \dots, [k/2]\}$, $r \in \{0, 1, \dots, k-2j\}$, exist. This result will be proved in Proposition 1.15.

Remarks.

(1.12) If $k = 1$ the space $\mathbb{L}^{1,2}_{1,c}$ coincides with the space introduced in Definition 1.8.

(1.13) Let $k = 2$ and $T = [0,1]$. The set $\mathbb{L}^{1,2}_{2,c}$ is the suitable space where the Stratonovich integral of order 2 can be defined (see Proposition 2.7 in Solé and Utzet, 1990). It will be widely used in this paper.

(1.14) If $X \in \mathbb{L}^{1,2}_{k,c(j,r)}$,

$$\operatorname*{ess\,sup}_{(t_{j+1}, \dots, t_{2j}, t_{2j+1}, \dots, t_{2j+r})} \|(D^{(i_1 \cdots i_r)}_{(h_1 \cdots h_j)} X)_{(t_{j+1}, \dots, t_{2j+r}, \cdot)}\|_{\mathbb{L}^{1,2}_{k-2j-r}} < \infty.$$

Next we prove an integral representation for the trace terms. This property allow us to state an iteration result which is on the basis of the Fubini theorem for the Stratonovich integral.

Proposition 1.15. *Let X be a process belonging to $\mathbb{L}^{1,2}_{k,c(j,r)}$. The trace $T_{j,r}(X)$ exists and, in addition,*

$$T_{j,r}(X) = \int_{T^{j+r}} (\nabla^r_j X)_{(t_{j+1}, \dots, t_{2j+r}, \cdot)} \mathrm{d}t_{j+1} \cdots \mathrm{d}t_{2j+r}. \tag{1.18}$$

For $k = 1$ and $T = [0,1]$ this result has been proved in Nualart and Pardoux (1988, Theorem 7.3). We need an extension to $T = [0,1]^2$, $k = 2$ and, consequently, for the traces $T_{0,1}(X)$, $T_{0,2}(X)$ and $T_{1,0}(X)$. The trace $T_{0,2}(X)$ has been considered in Solé and Utzet (1991, Theorem 4.7). The ideas of their proof can also be used in our general setting and provide a unified approach.

Proof of Proposition 1.15. Given an increasing sequence of partitions of T, $\{\pi_n, n \geq 1\}$, for $z = (s,t) \in \Delta^n_i = (s_i, s_{i+1}] \times (t_i, t_{i+1}]$ we denote by $\Delta^{n,++}_i(z)$ the subrectangle of Δ^n_i given by $(s, s_{i+1}] \times (t, t_{i+1}]$. We will omit the upper index n, to simplify the notation.

It holds that

$$\lim_{n \to \infty} \sum_{i_1, \dots, i_r \in \{1, \dots, r_n\}} \frac{|\Delta^{++}_{i_1}(z_1)| \cdots |\Delta^{++}_{i_r}(z_r)|}{|\Delta_{i_1}| \cdots |\Delta_{i_r}|} \mathbf{1}_{\Delta_{i_1}}(z_1) \cdots \mathbf{1}_{\Delta_{i_r}}(z_r) = \frac{1}{4^r},$$

in the $\sigma(L^1(T^r), L^\infty(T^r))$ topology, where $r_n = r_{\pi_n}$. Consequently, for any bounded stochastic process $Y = \{Y_{\underline{z}}, \underline{z} \in T^r\}$,

$$\lim_{n \to \infty} E\left(\left|\int_{T^r} \left(\sum_{i_1, \dots, i_r} \frac{|\Delta^{++}_{i_1}(z_1)| \cdots |\Delta^{++}_{i_r}(z_r)|}{|\Delta_{i_1}| \cdots |\Delta_{i_r}|} \mathbf{1}_{\Delta_{i_1}}(z_1) \cdots \mathbf{1}_{\Delta_{i_r}}(z_r) - \frac{1}{4^r}\right) Y_{\underline{z}} \mathrm{d}\underline{z}\right|^2\right) = 0. \tag{1.19}$$

By the definition of the traces $T_{j,r}(X)$, in order to check (1.18) it suffices to show that for any $l \in \{0, \dots, k-2j-r\}$,

$$\lim_{n\to\infty} D_h^l\Bigg(\sum_{i_1,\dots,i_{j+r}} \frac{1}{|\Delta_{i_1}|\cdots|\Delta_{i_{j+r}}|} \int_{\Delta_{i_1}^2\times\cdots\times\Delta_{i_{j+r}}^2} D_{\underline{s}}^r X_{\underline{t}}\, dt_1\cdots dt_{2j}\, ds_1\, dt_{2j+1}\cdots ds_r\, dt_{2j+r}$$

$$- \int_{T^{j+r}} (\nabla_j^r X)_{(t_{j+1},\dots,t_{2j+r},\cdot)}\, dt_{j+1}\cdots dt_{2j+r}\Bigg) = 0, \tag{1.20}$$

in $L^2(T^{k-2j-r+l}\times\Omega)$. We recall that X is a symmetric process.

The idea consists in decomposing the first integral in (1.20) according to the decomposition of $\nabla_j^r X$ given in (1.17). This procedure leads to 4^{j+r} terms. We will only consider one of them, since all are analogous. More precisely, our aim is to check

$$\lim_{n\to\infty} E(A_n) = 0, \tag{1.21}$$

where

$$A_n = \int_{T^{k-2j-r+l}} \Bigg[\int_{T^{j+r}} \Bigg[\sum_{i_1,\dots,i_{j+r}} \frac{1}{|\Delta_{i_1}|\cdots|\Delta_{i_{j+r}}|}$$

$$\times \Bigg(\int_{\Delta_{i_1}^{++}(t_1)\times\cdots\times\Delta_{i_j}^{++}(t_j)\times\Delta_{i_{j+1}}^{++}(s_1)\times\cdots\times\Delta_{i_{j+r}}^{++}(s_r)}$$

$$D_{\underline{u}}^l D_{\underline{s}}^r X_{\underline{t}}\, dt_{j+1}\cdots dt_{2j}\, dt_{2j+1}\cdots dt_{2j+r}\Bigg) 1_{\Delta_{i_1}}(t_1)\cdots 1_{\Delta_{i_j}}(t_j) 1_{\Delta_{i_{j+1}}}(s_1)\cdots 1_{\Delta_{i_{j+r}}}(s_r)$$

$$- \frac{1}{4^{j+r}} D_{\underline{u}}^l (D_{(1\cdots 1)}^{(1\cdots 1)} X)_{(t_1,\dots,t_j,s_1,\dots,s_r,\cdot)} \Bigg] dt_1\cdots dt_j\, ds_1\cdots ds_r \Bigg]^2 dt_{2j+r+1}\dots dt_k\, d\underline{u}.$$

Set

$$A_n^1 = \int_{T^{k-2j-r+l}} \Bigg[\int_{T^{j+r}} \sum_{i_1,\dots,i_{j+r}} \frac{|\Delta_{i_1}^{++}(t_1)|\cdots|\Delta_{i_j}^{++}(t_j)|\,|\Delta_{i_{j+1}}^{++}(s_1)|\cdots|\Delta_{i_{j+r}}^{++}(s_r)|}{|\Delta_{i_1}|\cdots|\Delta_{i_{j+r}}|}$$

$$\times 1_{\Delta_{i_1}}(t_1)\cdots 1_{\Delta_{i_j}}(t_j) 1_{\Delta_{i_{j+1}}}(s_1)\cdots 1_{\Delta_{i_{j+r}}}(s_r)$$

$$\times \Bigg(\frac{1}{|\Delta_{i_1}^{++}(t_1)|\cdots|\Delta_{i_j}^{++}(t_j)|\,|\Delta_{i_{j+1}}^{++}(s_1)|\cdots|\Delta_{i_{j+r}}^{++}(s_r)|}$$

$$\times \int_{\Delta_{i_1}^{++}(t_1)\times\cdots\times\Delta_{i_j}^{++}(t_j)\times\Delta_{i_{j+1}}^{++}(s_1)\times\cdots\times\Delta_{i_{j+r}}^{++}(s_r)}$$

$$\times D_{\underline{u}}^l D_{\underline{s}}^r X_{\underline{t}}\, dt_{j+1}\cdots dt_{2j}\, dt_{2j+1}\cdots dt_{2j+r}$$

$$- D_{\underline{u}}^l (D_{(1\cdots 1)}^{(1\cdots 1)} X)_{(t_1,\dots,t_j,s_1,\dots,s_r,\cdot)} \Bigg) dt_1\cdots dt_j\, ds_1\cdots ds_r \Bigg]^2 dt_{2j+r+1}\dots dt_k\, d\underline{u},$$

$$A_n^2 = \int_{T^{k-2j-r+l}} \Bigg[\int_{T^{j+r}} \Bigg\{ \sum_{i_1, \dots, i_{j+r}} \frac{|\Delta_{i_1}^{++}(t_1)| \cdots |\Delta_{i_j}^{++}(t_j)| \, |\Delta_{i_{j+1}}^{++}(s_1)| \cdots |\Delta_{i_{j+r}}^{++}(s_r)|}{|\Delta_{i_1}| \cdots |\Delta_{i_{j+r}}|}$$

$$1_{\Delta_{i_1}}(t_1) \cdots 1_{\Delta_{i_j}}(t_j) 1_{\Delta_{i_{j+1}}}(s_1) \cdots 1_{\Delta_{i_{j+r}}}(s_r) - \frac{1}{4^{j+r}} \Bigg\} D_{\underline{u}}^l (D^{(1\,\overset{r}{\cdots}\,1)}_{(1\,\overset{j}{\cdots}\,1)} X)_{(t_1, \dots, t_j, s_1, \dots, s_r, \cdot)}$$

$$\mathrm{d}t_1 \cdots \mathrm{d}t_j \mathrm{d}s_1 \cdots \mathrm{d}s_r \Bigg]^2 \mathrm{d}t_{2j+r+1} \cdots \mathrm{d}t_k \mathrm{d}\underline{u}.$$

We want to prove $\lim_{n \to \infty} E(A_n^i) = 0$ for $i = 1, 2$. Schwarz's inequality applied repeatedly yields

$$E(A_n^1) \leqslant \int_{T^{j+r}} \Bigg\{ \sum_{i_1, \dots, i_{j+r}} \frac{1}{|\Delta_{i_1}| \cdots |\Delta_{i_{j+r}}|} 1_{\Delta_{i_1}}(t_1) \cdots 1_{\Delta_{i_j}}(t_j) 1_{\Delta_{i_{j+1}}}(s_1) \cdots 1_{\Delta_{i_{j+r}}}(s_r)$$

$$\times E \Bigg[\int_{T^{k-2j-r+l}} \Bigg(\int_{\Delta_{i_1}^{++}(t_1) \times \cdots \times \Delta_{i_j}^{++}(t_j) \times \Delta_{i_{j+1}}^{++}(s_1) \times \cdots \times \Delta_{i_{j+r}}^{++}(s_r)}$$

$$\times \{ D_{\underline{u}}^l (D_{\underline{s}}^r X_{\underline{t}} - (D^{(1\,\overset{r}{\cdots}\,1)}_{(1\,\overset{j}{\cdots}\,1)} X)_{(t_1, \dots, t_j, s_1, \dots, s_r, \cdot)}) \}^2$$

$$\times \mathrm{d}t_{j+1} \cdots \mathrm{d}t_{2j} \mathrm{d}t_{2j+1} \cdots \mathrm{d}t_{2j+r} \Bigg) \mathrm{d}t_{2j+r+1} \dots \mathrm{d}t_k \mathrm{d}\underline{u} \Bigg] \Bigg\}$$

$$\times \mathrm{d}t_1 \cdots \mathrm{d}t_j \mathrm{d}s_1 \cdots \mathrm{d}s_r.$$

Using (1.16), the definition of the space $\mathbb{L}^{1,2}_{k,c(j,r)}$ (see Definition 1.11) and bounded convergence we obtain

$$\lim_{n \to \infty} E(A_n^1) = 0.$$

Property (1.19) and bounded convergence also yields

$$\lim_{n \to \infty} E(A_n^2) = 0.$$

Since $E(A_n) \leqslant 2E(A_n^1 + A_n^2)$, the proof of (1.21) is complete. □

The next definition is motivated by the problem of proving the iteration result on the traces. The idea is to introduce more uniformity on the spaces described in Definition 1.11 for $k = 2$.

Definition 1.16. We denote by $\mathbb{D}_u^{1,2}$ the space $\{F \in L^2(\Omega)\colon \operatorname{ess\,sup}_t \|D_t F\|_{L^2(\Omega)} < \infty\}$. Set $\|F\|_u^{1,2} := \|F\|_{L^2(\Omega)} + \operatorname{ess\,sup}_t \|D_t F\|_{L^2(\Omega)}$. We denote by $\mathbb{L}^{1,2}_{2,c(0,1),u}$ the set of processes X in $\mathbb{L}^{1,2}_{2,c(0,1)}$ such that

(1.16.1) $\quad \operatorname*{ess\,sup}_{t_1, t_2, s} \|D_s X_{t_1, t_2}\|_u^{1,2} < \infty,$

(1.16.2) there exists a neighborhood V of the set $\{(t_1, t_2, s) \in T^3\colon t_1 = s\}$ such that, for any $i \in \{1, 2, 3, 4\}$, there exists a version of DX with the property that the mapping $(t_1, t_2, s) \mapsto D_{t_1} X_{s, t_2}$ defined on $V \cap \{(t_1, t_2, s) \in T^3\colon s R^i t_1\}$ and taking values in $\mathbb{D}_u^{1,2}$, is continuous in the variable s uniformly in (t_1, t_2).

Note that, for $X \in \mathbb{L}^{1,2}_{2,c(0,1),u}$, one can define

$$((D^{(i)}X)_{(t_1,\cdot)})_{t_2} = \mathbb{D}^{1,2}_u - \lim D_{t_1} X_{s,t_2},$$

as $s \to t_1$, with $s R^i t_1$, uniformly in (t_1, t_2) and it holds that

$$\operatorname*{ess\,sup}_{t_1,t_2} \| ((D^{(i)}X)_{(t_1,\cdot)})_{t_2} \|^{1,2}_u < \infty.$$

Remarks

(**1.17**) Assume $X \in \mathbb{L}^{1,2}_{2,c(0,1),u}$; in this case $\sum_i (1/|\Delta_i|) \int_{\Delta_i^2} D_s X_{t_1,t_2} \mathrm{d}t_1 \, \mathrm{d}s$ converges to $\int_T ((\nabla^1_0 X)_{(t_1,\cdot)})_{t_2} \mathrm{d}t_1$ in $\mathbb{D}^{1,2}_u$ as $n \to \infty$, uniformly in t_2. This can be easily verified checking the details of the proof of (1.18) for $k = 2$, $j = 0$, $r = 1$, under this stronger hypothesis.

Furthermore,

(1.17.1) if $Y \in \mathbb{L}^{1,\infty}_2$ and $X \in \mathbb{L}^{1,2}_{2,c(0,1),u}$, then

$$\sum_i \frac{1}{|\Delta_i|} \int_{\Delta_i^2} D_s X_{t_1,t_2} Y_{t_1,h} \mathrm{d}s \, \mathrm{d}t_1$$

converges to $\int_T ((\nabla^1_0 X)_{(t_1,\cdot)})_{t_2} Y_{t_1,h} \mathrm{d}t_1$ in $\mathbb{D}^{1,2}_u$ as $n \to \infty$, uniformly in (h, t_2).

(**1.18**) A slight modification in the proof of Proposition 1.15 shows

(1.18.1) if $Y \in L^\infty(T \times \Omega)$ and $X \in \mathbb{L}^{1,2}_c$, then

$$\sum_i \frac{1}{|\Delta_i|} \int_{\Delta_i^2} D_s X_t Y_s \mathrm{d}s \, \mathrm{d}t \text{ converges to } \int_T (\nabla^1_0 X)_t Y_t \mathrm{d}s$$

in $L^2(\Omega)$ as $n \to \infty$,

(1.18.2) if $Y \in L^\infty(T^2 \times \Omega)$ and $X \in \mathbb{L}^{1,2}_{2,c(0,2)}$, then

$$\sum_{i,j} \frac{1}{|\Delta_i| \, |\Delta_j|} \int_{\Delta_i^2 \times \Delta_j^2} D^2_{s,r} X_{t_1,t_2} Y_{s,r} \mathrm{d}s \, \mathrm{d}t_1 \, \mathrm{d}r \, \mathrm{d}t_2$$

converges to $\int_{T^2} (\nabla^2_0 X)_{(s,r)} Y_{s,r} \mathrm{d}s \, \mathrm{d}r$ in $L^2(\Omega)$ as $n \to \infty$.

Now we can state the property of the iteration of the traces.

Proposition 1.19. *Let $X \in \mathbb{L}^{1,2}_{2,c(0,1),u} \cap \mathbb{L}^{1,2}_{2,c(0,2)}$ be a process such that $T_{0,1}(X) \in \mathbb{L}^{1,2}_c$. Then, $T_{0,2}(X)$ and $T_{0,1}(T_{0,1}(X))$ exist and coincide.*

Proof. By Proposition 1.15 we know that the traces $T_{0,1}(X)$, $T_{0,2}(X)$ and $T_{0,1}(T_{0,1}(X))$ exist. In order to check that $T_{0,1}(T_{0,1}(X)) = T_{0,2}(X)$, we will show that for any $i, j \in \{1, 2, 3, 4\}$,

$$\int_T \left(D^{(i)} \left(\int_T (D^{(j)} X)_{(t_1,\cdot)} \mathrm{d}t_1 \right) \right)_{t_2} \mathrm{d}t_2 = \int_{T^2} (D^{(i,j)} X)_{t_1,t_2} \mathrm{d}t_1 \, \mathrm{d}t_2. \tag{1.22}$$

By (1.16), the left-hand side of (1.22) is

$$\int_T \left(\lim_{s \to t_2} D_{t_2} \left(\int_T ((D^{(j)} X)_{(t_1, \cdot)})_s \mathrm{d}t_1 \right) \right) \mathrm{d}t_2 \tag{1.23}$$

where the limit is in the $L^2(\Omega)$ sense, and $s R^i t_2$. Therefore, it suffices to prove that, for a.e. $t \in T$,

$$C(s,t) = \left\| \int_T (D_t((D^{(j)} X)_{(t_1, \cdot)})_s - (D^{(i,j)} X)_{t_1, t}) \mathrm{d}t_1 \right\|_{L^2(\Omega)}$$

converges to 0 as $s \to t, s R^i t$.

Notice that

$$C(s,t) \leqslant \operatorname*{ess\,sup}_{t_1} \| D_t((D^{(j)} X)_{(t_1, \cdot)})_s - (D^{(i,j)} X)_{t_1, t} \|_{L^2(\Omega)}.$$

The second term of this inequality is bounded by $C_1(s,t,r) + C_2(s,t,r)$, $r \in T$, where

$$C_1(s,t,r) = \operatorname*{ess\,sup}_{t_1} \| D_t[((D^{(j)} X)_{(t_1, \cdot)})_s - D_{t_1} X_{r,s}] \|_{L^2(\Omega)}$$

and

$$C_2(s,t,r) = \operatorname*{ess\,sup}_{t_1} \| D^2_{t_1, t} X_{r,s} - (D^{(i,j)} X)_{t_1, t} \|_{L^2(\Omega)}.$$

Fix $\varepsilon > 0$. Since $X \in \mathbb{L}^{1,2}_{2,c(0,2)}$, there exist $\eta > 0$, only depending on ε, such that for any $(r,s) \in T^2$, $|r - t_1| + |s - t| < \eta$, $r R^j t_1$, $s R^i t$, we have $\operatorname{ess\,sup}_t C_2(s,t,r) < \varepsilon/2$. Analogously, the assumption $X \in \mathbb{L}^{1,2}_{2,c(0,1),u}$ yields the existence of $\delta > 0$, only depending on ε, such that for any $r \in T$, $|r - t_1| < \delta$, $r R^j t_1$, $\operatorname{ess\,sup}_{s,t} C_1(s,t,r) < \varepsilon/2$. Then, taking $r \in T$ with $|r - t_1| < \inf(\delta, \eta)$, $r R^j t_1$, we obtain, for any $s \in T$ with $|s - t| < \eta$, $s R^i t$, $C(s,t) < \varepsilon$. This completes the proof of the proposition, because ε is arbitrary. $\square$

We will finish this section by proving a formula for the trace of a Skorohod integral, but first we will establish a result on commutation between Skorohod's and Lebesgue's integral (see also Solé and Utzet, 1991, Lemma 3.12, Thieullen, 1991, Theorem 2.8).

Lemma 1.20. *Let X be a symmetric process belonging to $L^2(T^k \times \Omega)$. Assume that for any fixed $(r_1, \ldots, r_{k-1}) \in T^{k-1}$ the process $X_{(r_1, \ldots, r_{k-1}, \cdot)} \in Dom\,\delta$, $\delta[X_{(r_1, \ldots, r_{k-1}, \cdot)}]$ is Lebesgue integrable and $\int_{T^{k-1}} \delta[X_{(r_1, \ldots, r_{k-1}, \cdot)} \mathrm{d}r_1 \cdots \mathrm{d}r_{k-1} \in L^2(\Omega)$. Then, $\int_{T^{k-1}} X_{(r_1, \ldots, r_{k-1}, \cdot)} \mathrm{d}r_1 \cdots \mathrm{d}r_{k-1} \in Dom\,\delta$ and*

$$\delta\left(\int_{T^{k-1}} X_{(r_1, \ldots, r_{k-1}, \cdot)} \mathrm{d}r_1 \cdots \mathrm{d}r_{k-1} \right) = \int_{T^{k-1}} \delta(X_{(r_1, \ldots, r_{k-1}, \cdot)}) \mathrm{d}r_1 \cdots \mathrm{d}r_{k-1}. \tag{1.24}$$

Proof. Let $F \in \mathbb{D}^{1,2}$. The classical Fubini's theorem and the duality relation between δ and D yields

$$\begin{aligned} E&\left\{F\int_{T^{k-1}}\delta(X_{(r_1,\ldots,r_{k-1},\cdot)})\mathrm{d}r_1\cdots\mathrm{d}r_{k-1}\right\}\\ &=\int_{T^{k-1}}E\{F\delta(X_{(r_1,\ldots,r_{k-1},\cdot)})\}\,\mathrm{d}r_1\cdots\mathrm{d}r_{k-1}\\ &=E\left\{\int_{T^k}D_sF\,X_{(r_1,\ldots,r_{k-1},s)}\mathrm{d}r_1\cdots\mathrm{d}r_{k-1}\,\mathrm{d}s\right\}\\ &=E\left\{\int_T D_sF\int_{T^{k-1}}X_{(r_1,\ldots,r_{k-1},s)}\mathrm{d}r_1\cdots\mathrm{d}r_{k-1}\,\mathrm{d}s\right\}. \end{aligned}$$

This proves that $\int_{T^{k-1}}X_{(r_1,\ldots,r_{k-1},\cdot)}\mathrm{d}r_1\cdots\mathrm{d}r_{k-1}\in Dom\,\delta$ and that (1.24) is satisfied. □

Proposition 1.21. *Let X be a process in $\mathbb{L}^{1,2}_{2,c(0,1)}\cap L^{1,2}_{2,c(1,0)}$. Then*

$$T_{0,1}(\delta(X))=\delta(T_{0,1}(X))+T_{1,0}(X), \tag{1.25}$$

in the sense that all terms appearing in (1.25) *exist and the relation* (1.25) *holds.*

Proof. Since $X\in\mathbb{L}^{1,2}_{2,c(0,1)}$, the trace $T_{0,1}(X)$ exists and belongs to $\mathbb{L}^{1,2}$. Furthermore,

$$T_{0,1}(X)=\mathbb{L}^{1,2}-\lim_{n\to\infty}\sum_i\frac{1}{|\Delta_i|}\int_{\Delta_i^2}D_sX_{t,\cdot}\,\mathrm{d}s\,\mathrm{d}t. \tag{1.26}$$

Thus $\delta(T_{0,1}(X))$ exists. It is also clear that $T_{1,0}(X)$ exists as well as $\delta(X_{t,\cdot})$ for any $t\in T$, a.e. The rules of the anticipating calculus yield

$$\sum_i\frac{1}{|\Delta_i|}\int_{\Delta_i^2}D_s(\delta(X_{t,\cdot}))\mathrm{d}s\,\mathrm{d}t=\sum_i\frac{1}{|\Delta_i|}\int_{\Delta_i^2}\{\delta(D_sX_{t,\cdot})+X_{t,s}\}\,\mathrm{d}s\,\mathrm{d}t.$$

Lemma 1.20 applied to the process $\{D_sX_{t,r},(s,t,r)\in T^3\}$ and (1.26) ensure

$$L^2(\Omega)-\lim_{n\to\infty}\sum_i\frac{1}{|\Delta_i|}\int_{\Delta_i^2}\delta(D_sX_{t,\cdot}))\mathrm{d}s\,\mathrm{d}t=\delta(T_{0,1}(X)).$$

Since

$$T_{1,0}(X)=L^2-\lim_{n\to\infty}\sum_i\frac{1}{|\Delta_i|}\int_{\Delta_i^2}X_{t,s}\mathrm{d}t\,\mathrm{d}s,$$

we conclude

$$L^2-\lim_{n\to\infty}\sum_i\frac{1}{|\Delta_i|}\int_{\Delta_i^2}D_s(\delta(X_{t,\cdot}))\mathrm{d}s\,\mathrm{d}t=\delta(T_{0,1}(X))+T_{1,0}(X),$$

proving (1.25). □

2. A Green formula for rectangles

In this section we prove the main result of this paper: a Green formula for anticipating processes with respect to nonadapted semimartingales U (see (0.1)). In order to avoid some unnecessary technicalities all the proofs are done in the particular case $v \equiv 0$. Indeed, the interesting problems are produced by the stochastic integrals.

Section 2.1 is devoted to proving a Hu–Meyer formula for second-order integrals. That means, we need an extension of the result proved in Delgado and Sanz-Solé (1992) for $k = 2$. The generalization from W to U as integrator is by no means straightforward. Indeed, one important element underlying the proof of the Hu–Meyer formula is the Wiener chaos decomposition of the integrator, which is almost trivial for W and not at all for U. Instead of dealing with products of Itô's multiple integrals we encounter products of Skorohod integrals; hence calculus is much harder. In section 2.2 we prove Fubini's theorem for the Stratonovich integral, it is obtained by combining Hu–Meyer's formula and the property on iteration of traces proved in Proposition 1.19. Sections 2.3 and 2.4 are devoted to the proof of Green's formula and applications.

2.1. A Hu–Meyer type formula

Our aim now is to state a relation between the k-Stratonovich and Skorohod integrals with respect to U for $k = 2$; this result is given in Proposition 2.1.5. The case $k = 1$ has been studied in Proposition 1.9 (see (1.11)). Lemma 2.1.3 is a simplified version of the main result. In fact, the integrator in this lemma is the process given by the Skorohod integral $\int_{u_r} \mathrm{d}W_r$. The analysis of the extra contribution of U, that means $\int (\nabla_0^1 u)_r \,\mathrm{d}r$, is carried out in Proposition 2.1.5.

We start by proving some technical convergences to be used in the sequel.

Lemma 2.1.1. *Let $X \in \mathbb{L}_2^{1,2}$ and $Y \in \mathbb{L}_2^{1,\infty}$. Then*

$$\sum_{i,j} \frac{1}{|\Delta_i|\,|\Delta_j|}\left(\int_{\Delta_i \times \Delta_j} X_z \,\mathrm{d}\underline{z}\right) Y_{r,s} 1_{\Delta_i}(r) 1_{\Delta_j}(s) \xrightarrow[n\to\infty]{} X_{r,s} Y_{r,s} \tag{2.1}$$

in $\mathbb{L}_2^{1,2}$. In particular,

$$L^2 - \lim_{n\to\infty} \sum_{i,j} \frac{1}{|\Delta_i|\,|\Delta_j|}\left(\int_{\Delta_i \times \Delta_j} X_z \,\mathrm{d}\underline{z}\right) \int_{\Delta_i \times \Delta_j} Y_{r,s} \,\mathrm{d}r\,\mathrm{d}s = \int_{T^2} X_{r,s} Y_{r,s} \,\mathrm{d}r\,\mathrm{d}s, \tag{2.2}$$

and

$$\mathbb{L}^{1,2} - \lim_{n\to\infty} \int_T \sum_{i,j} \frac{1}{|\Delta_i|\,|\Delta_j|}\left(\int_{\Delta_i \times \Delta_j} X_z \,\mathrm{d}\underline{z}\right) Y_{r,s} 1_{\Delta_i}(r) 1_{\Delta_j}(s) \,\mathrm{d}s = \int_T X_{r,s} Y_{r,s} \,\mathrm{d}s. \tag{2.3}$$

Proof. The convergence (2.1) is an easy corollary of Proposition 3.5 in Jolis and Sanz-Solé (1990), due to the boundedness properties of the processes Y. The convergences (2.2) and (2.3) follow clearly from (2.1). □

Lemma 2.1.2. *Let* $X \in \mathbb{L}^{1,2}_{2,c(0,1),u}$ *and* $Y = \{Y_{r,s}, (r,s) \in T^2\}$ *be a process belonging to* $\mathbb{L}^{1,\infty}_2$. *Then*

$$\int_T \sum_{i,j} \frac{1}{|\Delta_i|\,|\Delta_j|}\left(\int_{\Delta_i \times \Delta_j} D_r X_{\underline{z}}\, d\underline{z}\right) Y_{r,s} 1_{\Delta_i}(r) 1_{\Delta_j}(s)\, dr \xrightarrow[n\to\infty]{} \int_T ((\nabla_0^1 X)_{(r,\cdot)})_s Y_{r,s}\, dr, \tag{2.4}$$

in $\mathbb{L}^{1,2}$. *In particular,*

$$\sum_{i,j} \frac{1}{|\Delta_i|\,|\Delta_j|}\left(\int_{\Delta_i \times \Delta_j} D_r X_{\underline{z}}\, d\underline{z}\right) \int_{\Delta_i \times \Delta_j} Y_{r,s}\, dr\, ds \xrightarrow[n\to\infty]{} \int_{T^2} ((\nabla_0^1 X)_{(r,\cdot)})_s Y_{r,s}\, dr\, ds, \tag{2.5}$$

in $L^2(\Omega)$.

Proof. Let us first check the convergence (2.4) in $L^2(T \times \Omega)$. The corresponding convergence for the first-order derivatives is proved using the same arguments. We have

$$E\int_T ds \left| \int_T \sum_{i,j} \frac{1}{|\Delta_i|\,|\Delta_j|}\left(\int_{\Delta_i \times \Delta_j} D_r X_{\underline{z}}\, d\underline{z}\right) Y_{r,s} 1_{\Delta_i}(r) 1_{\Delta_j}(s)\, dr \right.$$
$$\left. - \int_T ((\nabla_0^1 X)_{(r,\cdot)})_s Y_{r,s}\, dr \right|^2 \leqslant 2(A_n + B_n),$$

where

$$A_n = E\int_T ds \left| \sum_j \frac{1}{|\Delta_j|}\left(\int_{\Delta_j} \left\{ \sum_i \frac{1}{|\Delta_i|} \int_{\Delta_i^2} D_r X_{\underline{z}} Y_{r,s}\, dz_1\, dr \right.\right.\right.$$
$$\left.\left.\left. - \int_T ((\nabla_0^1 X)_{(r,\cdot)})_{z_2} Y_{r,s}\, dr \right\} dz_2 \right) 1_{\Delta_j}(s) \right|^2$$

and

$$B_n = E\int_T ds \left| \sum_j \frac{1}{|\Delta_j|}\left(\int_{\Delta_j} dz_2 \int_T ((\nabla_0^1 X)_{(r,\cdot)})_{z_2} Y_{r,s}\, dr\right) 1_{\Delta_j}(s) \right.$$
$$\left. - \int_T ((\nabla_0^1 X)_{(r,\cdot)})_s Y_{r,s}\, dr \right|^2,$$

$\underline{z} = (z_1, z_2)$.

The boundedness hypotheses on Y ensure that

$$L^2 - \lim_{n\to\infty} \sum_i \frac{1}{|\Delta_i|} \int_{\Delta_i^2} D_r X_{\underline{z}} Y_{r,s}\, dz_1\, dr = \int_T ((\nabla_0^1 X)_{(r,\cdot)})_{z_2} Y_{r,s}\, dr.$$

uniformly in the variables z_2 and s (see Remark (1.17.1)). Hence, by Schwarz's inequality

$$A_n \leqslant \int_T \mathrm{d}s \sum_i \frac{1}{|\Delta_j|}\left(\int_{\Delta_j} E\left|\sum_i \frac{1}{|\Delta_i|}\int_{\Delta_i^2} D_r X_{\underline{z}} Y_{r,s}\,\mathrm{d}z_1\,\mathrm{d}r \right.\right.$$

$$\left.\left. - \int_T ((\nabla_0^1 X)_{(r,\cdot)})_{z_2} Y_{r,s}\,\mathrm{d}r\right|^2 \mathrm{d}z_2\right) 1_{\Delta_j}(s)$$

$$\leqslant \sup_{z_2,s} E\left|\sum_i \frac{1}{|\Delta_i|}\int_{\Delta_i^2} D_r X_{\underline{z}} Y_{r,s}\,\mathrm{d}z_1\,\mathrm{d}r \right.$$

$$\left. - \int_T ((\nabla_0^1 X)_{(r,\cdot)})_{z_2} Y_{r,s}\,\mathrm{d}r\right|^2 \xrightarrow[n\to\infty]{} 0.$$

In order to prove that $\lim_{n\to\infty} B_n = 0$, it suffices to see that

$$L^2(T^2\times\Omega) - \lim_{n\to\infty} \sum_j \frac{1}{|\Delta_j|}\left(\int_{\Delta_j} ((\nabla_0^1 X)_{(r,\cdot)})_{z_2}\,\mathrm{d}z_2\right) Y_{r,s} 1_{\Delta_j}(s)$$

$$= ((\nabla_0^1 X)_{(r,.)})_s Y_{r,s}.$$

Since Y is bounded, this follows from Proposition 3.5 in Jolis and Sanz-Solé (1990). Consequently, the convergence (2.4) in $L^2(T\times\Omega)$ is established. The convergence (2.5) clearly follows from (2.4). □

Lemma 2.1.3. *Let* $X \in \mathbb{L}^{1,2}_{2,c(0,1),u} \cap \mathbb{L}^{1,2}_{2,c(0,2)}$ *and* $u \in \mathbb{L}^{2,\infty}$. *Set*

$$A^n = \sum_{i,j} \frac{1}{|\Delta_i||\Delta_j|}\left(\int_{\Delta_i\times\Delta_j} X_{\underline{z}}\,\mathrm{d}\underline{z}\right)\delta(1_{\Delta_i}\cdot u)\delta(1_{\Delta_j}\cdot u)$$

and

$$B^n = \sum_i \frac{1}{|\Delta_i|^2}\left(\int_{\Delta_i^2} X_{\underline{z}}\,\mathrm{d}\underline{z}\right)\int_{\Delta_i} u_r^2\,\mathrm{d}r. \tag{2.6}$$

Then, $L^2 - \lim_{n\to\infty} A^n$ *exists if and only if* $L^2 - \lim_{n\to\infty} B^n$ *exists, and in this case*

$$\begin{aligned} L^2 - \lim_{n\to\infty} A^n &= \delta^2(X_{r,s}u_r u_s) + 2\int_T \delta(X_{r,s}u_r D_r u_s)\,\mathrm{d}r \\ &\quad + \int_{T^2} X_{r,s} D_r u_s D_s u_r\,\mathrm{d}r\,\mathrm{d}s + 2\delta\left\{\int_T ((\nabla_0^1 X)_{(\cdot,s)})_r u_r u_s\,\mathrm{d}s\right\} \\ &\quad + \int_{T^2} (\nabla_0^2 X)_{r,s} u_r u_s\,\mathrm{d}r\,ds \\ &\quad + 2\int_{T^2} ((\nabla_0^1 X)_{(\cdot,s)})_r u_r D_r u_s\,\mathrm{d}r\,\mathrm{d}s + L^2 - \lim_{n\to\infty} B^n. \end{aligned} \tag{2.7}$$

Set

$$(Xuu)_{r,s} = X_{r,s}u_r u_s. \tag{2.8}$$

In Lemma 2.1.4 we prove that, under additional conditions on X and u,

$$L^2 - \lim_{n\to\infty} B_n = T_{1,0}(Xuu) = \int_T \nabla_1^0(Xuu)_s \, ds.$$

Proof of Lemma 2.1.3. Using Propositions 2.9 and 2.8 of Nualart and Zakai (1988), together with Proposition 3.4 of Nualart and Pardoux (1988) (which provide formulas for the product of two simple Skorohod integrals, the product of a random variable and a double Skorohod integral and that of a random variable and a simple Skorohod integral, respectively), we can transform A^n into the following expression:

$$\begin{aligned}
A^n = \sum_{i,j} \frac{1}{|\Delta_i||\Delta_j|} \Bigg\{ & \delta^2\left[\left(\int_{\Delta_i\times\Delta_j} X_{\underline{z}}\, d\underline{z}\right) 1_{\Delta_i}(r) 1_{\Delta_j}(s) u_r u_s\right] \\
& + 2\int_T \delta\left[\left(\int_{\Delta_i\times\Delta_j} X_{\underline{z}}\, d\underline{z}\right) 1_{\Delta_i}(r) 1_{\Delta_j}(s) u_r D_r u_s\right] dr \\
& + 2\int_T D_s\left(\int_{\Delta_i\times\Delta_j} X_{\underline{z}}\, d\underline{z}\right) \delta(1_{\Delta_i}(r) 1_{\Delta_j}(s) u_r u_s)\, ds \\
& - \int_{T^2} D^2_{r,s}\left(\int_{\Delta_i\times\Delta_j} X_{\underline{z}}\, d\underline{z}\right) 1_{\Delta_i}(r) 1_{\Delta_j}(s) u_r u_s\, dr\, ds \\
& - \int_{T^2} \left(\int_{\Delta_i\times\Delta_j} X_{\underline{z}}\, d\underline{z}\right) 1_{\Delta_i}(r) 1_{\Delta_j}(s) D_s u_r D_r u_s\, dr\, ds \\
& + 2\int_{T^2} 1_{\Delta_j}(s) D_r u_s D_s\left[\left(\int_{\Delta_i\times\Delta_j} X_{\underline{z}}\, d\underline{z}\right) 1_{\Delta_i}(r) u_r\right] dr\, ds \\
& + \int_T \left(\int_{\Delta_i\times\Delta_j} X_{\underline{z}}\, d\underline{z}\right) 1_{\Delta_i}(r) 1_{\Delta_j}(r) u_r^2\, dr \Bigg\}.
\end{aligned}$$

Set

$$A_1^n = \sum_{i,j} \frac{1}{|\Delta_i||\Delta_j|} \int_T \delta\left[\left(\int_{\Delta_i\times\Delta_j} D_s X_{\underline{z}}\, d\underline{z}\right) 1_{\Delta_i}(r) 1_{\Delta_j}(s) u_r u_s\right] ds,$$

$$A_2^n = \sum_{i,j} \frac{1}{|\Delta_i||\Delta_j|} \int_{T^2} \left(\int_{\Delta_i\times\Delta_j} D^2_{r,s} X_{\underline{z}}\, d\underline{z}\right) 1_{\Delta_i}(r) 1_{\Delta_j}(s) u_r u_s\, dr\, ds,$$

$$A_3^n = \sum_{i,j} \frac{1}{|\Delta_i||\Delta_j|} \int_{T^2} \left(\int_{\Delta_i\times\Delta_j} D_s X_{\underline{z}}\, d\underline{z}\right) 1_{\Delta_i}(r) 1_{\Delta_j}(s) D_r u_s u_r\, dr\, ds.$$

The convergences (2.1) with $Y_{r,s} = u_r u_s$, (2.3) with $Y_{r,s} = u_r D_r u_s$ and (2.2) with $Y_{r,s} = D_s u_r D_r u_s$, respectively, ensure

$$\begin{aligned}
L^2 - \lim_{n\to\infty} A^n = {} & \delta^2(X_{r,s} u_r u_s) + 2\int_T \delta(X_{r,\cdot} u_r D_r u_\cdot)\, dr + \int_{T^2} X_{r,s} D_s u_r D_r u_s\, dr\, ds \\
& + L^2 - \lim_{n\to\infty} (2A_1^n + A_2^n + 2A_3^n + B^n). \qquad (2.9)
\end{aligned}$$

Lemma 1.20 applied to the process $\{(\int_{\Delta_i\times\Delta_j} D_s X_{\underline{z}}\,d\underline{z})u_r u_s\, 1_{\Delta_i}(r)\, 1_{\Delta_j}(s), (r,s)\in T^2\}$ and then Lemma 2.1.2 (see (2.4)) with $Y_{r,s}=u_r u_s$ yield

$$L^2-\lim_{n\to\infty} A_1^n = \delta\left[\int_T ((\nabla_0^1 X)_{(\cdot,s)})_r u_r u_s\,ds\right]. \tag{2.10}$$

Remark 1.18.2 ensures

$$L^2-\lim_{n\to\infty} A_2^n = \int_{T^2} (\nabla_0^2 X)_{r,s} u_r u_s\,dr\,ds, \tag{2.11}$$

and (2.5) with $Y_{r,s}=u_r D_r u_s$ implies

$$L^2-\lim_{n\to\infty} A_3^n = \int_{T^2} ((\nabla_0^1 X)_{(\cdot,s)})_r u_r D_r u_s\,dr\,ds. \tag{2.12}$$

Thus, the existence of $L^2-\lim_{n\to\infty} A^n$ is equivalent to that of $L^2-\lim_{n\to\infty} B^n$. In addition, the equalities (2.9)–(2.12) show (2.7). □

Lemma 2.1.4. *Let $X\in L^2(T^2\times\Omega)$ be symmetric and such that* $\operatorname{ess\,sup}_{t_1,t_2}\|X_{t_1,t_2}\|_{L^4(\Omega)} < \infty$. *Assume $u\in L^\infty(T\times\Omega)$, the process Xuu, defined in* (2.8), *belongs to $\mathbb{L}^{1,2}_{2,c(1,0)}$ and there exists a version of u continuous in L^2. Then,*

$$L^2-\lim B^n := L^2-\lim_{n\to\infty}\sum_i \frac{1}{|\Delta_i|^2}\left(\int_{\Delta_i^2} X_{\underline{z}}\,d\underline{z}\right)\int_{\Delta_i} u_r^2\,dr = T_{1,0}(Xuu). \tag{2.13}$$

Proof. We use the same ideas and notation as for the proof of Proposition 1.15. First, we notice that

$$T_{1,0}(Xuu) = \int_T (\nabla_0^1(Xuu))_r\,dr.$$

We will check

$$\lim_{n\to\infty} D_n = 0, \tag{2.14}$$

where

$$D_n = E\Bigg(\int_T\Bigg[\sum_i \frac{1}{|\Delta_i|}\left(\int_{\Delta_i^{++}(z_1)} X_{z_1,z_2}\left(\frac{1}{|\Delta_i|}\int_{\Delta_i} u_r^2\,dr\right)dz_2\right)$$
$$\times 1_{\Delta_i}(z_1) - \tfrac{1}{4}(D_{(1)}(Xuu))_{z_1}\Bigg]dz_1\Bigg)^2.$$

This is one typical term which appears in the proof of (2.13). We have

$$D_n \leqslant 2(D_n^1 + D_n^2),$$

with

$$D_n^1 = E\left[\int_T \sum_i \frac{|\Delta_i^{++}(z_1)|}{|\Delta_i|} 1_{\Delta_i}(z_1)\left\{\frac{1}{|\Delta_i^{++}(z_1)|}\int_{\Delta_i^{++}(z_1)} [X_{z_1,z_2}\left(\frac{1}{|\Delta_i|}\int_{\Delta_i} u_r^2\,\mathrm{d}r\right)\right.\right.$$

$$\left.\left. - (D_{(1)}(Xuu))_{z_1}]\,\mathrm{d}z_2\right\}\mathrm{d}z_1\right]^2,$$

and

$$D_n^2 = E\left[\int_T \left\{\sum_i \frac{|\Delta_i^{++}(z_1)|}{|\Delta_i|} 1_{\Delta_i}(z_1) - \tfrac{1}{4}\right\}(D_{(1)}(Xuu))_{z_1}\,\mathrm{d}z_1\right]^2.$$

Clearly, $\lim_{n\to\infty} D_n^2 = 0$. Schwarz's inequality yields

$$D_n^1 \leqslant \int_T \mathrm{d}z_1\left(\sum_i \frac{1}{|\Delta_i|} 1_{\Delta_i}(z_1)\int_{\Delta_i^{++}(z_1)} \mathrm{d}z_2\, E\{X_{z_1,z_2}\left(\frac{1}{|\Delta_i|}\int_{\Delta_i} u_r^2\,\mathrm{d}r\right) - D_{(1)}(Xuu))_{z_1}\}^2\,\mathrm{d}z_2\right).$$

For any $z_1, z_2 \in T$ we can write

$$E\left(\left|X_{z_1,z_2}\left(\frac{1}{|\Delta_i|}\int_{\Delta_i} u_r^2\,\mathrm{d}r\right) - (D_{(1)}(Xuu))_{z_1}\right|^2\right)$$

$$\leqslant 2\left\{E\left|(Xuu)_{z_1,z_2} - X_{z_1,z_2}\left(\frac{1}{|\Delta_i|}\int_{\Delta_i} u_r^2\,\mathrm{d}r\right)\right|^2\right.$$

$$\left. + E|(Xuu)_{z_1,z_2} - (D_{(1)}(Xuu))_{z_1}|^2\right\}. \tag{2.15}$$

Fix $\varepsilon > 0$; since Xuu belongs to $\mathbb{L}^{1;2}_{2,c(1,0)}$, there exists a natural number n_0 such that for any $n \geqslant n_0$, and $z_1, z_2 \in \Delta_i$, $z_2 R^1 z_1$, the last term in (2.15) is less than or equal to ε. Furthermore, by Hölder's inequality

$$E\left(\left|(Xuu)_{z_1,z_2} - X_{z_1,z_2}\left(\frac{1}{|\Delta_i|}\int_{\Delta_i} u_r^2\,\mathrm{d}r\right)\right|^2\right)$$

$$\leqslant C \operatorname*{ess\,sup}_{z_1,z_2}(E|X_{z_1,z_2}|^4)^{1/4} \sup_i \frac{1}{|\Delta_i|}\int_{\Delta_i} (E(|u_{z_1}u_{z_2} - u_r^2|^4))^{1/4}\,\mathrm{d}r \xrightarrow[n\to\infty]{} 0.$$

Consequently (2.14) is proved. □

Assume $U = W$, that means, $u \equiv 1$. Theorem 3.1 of Delgado and Sanz-Solé (1992) yields

$$I_2^s(X) := I_2^{s,W}(X) = \delta^2(X) + 2\delta(T_{0,1}(X)) + T_{0,2}(X) + T_{1,0}(X), \tag{2.16}$$

which coincides with (2.7). Unfortunately the compact form of (2.16) does not seem to have a simple translation in our more general situation. For $(j, r) = (0, 0)$, there is no contraction of indices; the corresponding term in (2.16) is $\delta^2(X)$ and in (2.7) is

$$\delta^2(Xuu) + 2\int_T \delta(X_{r,s}u_r D_r u_s)\,\mathrm{d}r + \int_{T^2} X_{r,s} D_r u_s D_s u_r\,\mathrm{d}r\,\mathrm{d}s.$$

If $(j,r)=(1,0)$, there is a contraction between indices of X; for $u\equiv 1$ we obtain $T_{1,0}(X)$, in (2.7) we get $T_{1,0}(Xuu)$. Analogously, if $(j,r)=(0,2)$ there are two contractions for the two indices of the second derivative of X; this produces the term $T_{0,2}(X)$ in (2.16), which can also be written as $\int_{T^2}(\nabla_0^2 X)_{r,s}\,\mathrm{d}r\,\mathrm{d}s$; in (2.7) we obtain $\int_{T^2}(\nabla_0^2 X)_{r,s}u_r u_s\,\mathrm{d}r\,\mathrm{d}s$. The remaining terms in (2.16) and (2.7), respectively, correspond to $(j,r)=(0,1)$.

Proposition 2.1.5. *Let $X\in\mathbb{L}^{1,2}_{2,c(0,1),u}\cap\mathbb{L}^{1,2}_{2,c(0,2)}$, $u\in\mathbb{L}^{1,2}_c\cap\mathbb{L}^{2,\infty}$. Assume $\nabla_0^1 u\in\mathbb{L}^{1,\infty}$ and the existence of $L^2-\lim_{n\to\infty}B_n$ (see (2.6)). Then $X\in Dom\, I_2^{s,U}$ and*

$$\begin{aligned}I_2^{s,U}(X)&=\delta_2^U(X)+\delta\left(\int_T X_{r,s}u_sD_su_r\,\mathrm{d}s\right)+\int_{T^2}X_{r,s}D_ru_sD_su_r\,\mathrm{d}s\,\mathrm{d}r\\&\quad+2\delta\left(\int_T((\nabla_0^1X)_{(\cdot,s)})_ru_su_r\,\mathrm{d}s\right)+\int_{T^2}(\nabla_0^2X)_{r,s}u_su_r\,\mathrm{d}s\,\mathrm{d}r\\&\quad+2\int_{T^2}((\nabla_0^1X)_{(\cdot,s)})_rD_ru_su_r\,\mathrm{d}s\,\mathrm{d}r+\int_{T^2}X_{r,s}u_rD_r(\nabla_0^1u)_s\,\mathrm{d}r\,\mathrm{d}s\\&\quad+2\int_{T^2}((\nabla_0^1X)_{(\cdot,r)})_su_r(\nabla_0^1u)_s\,\mathrm{d}r\,\mathrm{d}s\to L^2-\lim_{n\to\infty}B_n.\end{aligned}\tag{2.17}$$

Proof. The hypotheses on u clearly yield $X\in Dom\,\delta_2^U$, since $X\in\mathbb{L}_2^{1,2}$. Then, we have to check that

$$L^2-\lim_{n\to\infty}\sum_{i,j}\frac{1}{|\Delta_i||\Delta_j|}\left(\int_{\Delta_i\times\Delta_j}X_{\underline{z}}\,\mathrm{d}\underline{z}\right)U(\Delta_i)U(\Delta_j)$$

exists and equals the right-hand side of (2.17). Set

$$\begin{aligned}A_1&=\sum_{i,j}\frac{1}{|\Delta_i||\Delta_j|}\left(\int_{\Delta_i\times\Delta_j}X_{\underline{z}}\,\mathrm{d}\underline{z}\right)\delta(1_{\Delta_i}\cdot u)\delta(1_{\Delta_j}\cdot u),\\A_2&=\sum_{i,j}\frac{1}{|\Delta_i||\Delta_j|}\left(\int_{\Delta_i\times\Delta_j}X_{\underline{z}}\,\mathrm{d}\underline{z}\right)\delta(1_{\Delta_i}\cdot u)\int_{\Delta_j}(\nabla_0^1u)_r\,\mathrm{d}r,\\A_3&=\sum_{i,j}\frac{1}{|\Delta_i||\Delta_j|}\left(\int_{\Delta_i\times\Delta_j}X_{\underline{z}}\,\mathrm{d}\underline{z}\right)\left(\int_{\Delta_i}(\nabla_0^1u)_r\,\mathrm{d}r\right)\left(\int_{\Delta_j}(\nabla_0^1u)_r\,\mathrm{d}r\right).\end{aligned}$$

Since

$$U(\Delta)=\delta(1_\Delta\cdot u)+\int_\Delta(\nabla_0^1u)_r\,\mathrm{d}r,$$

Clearly

$$\sum_{i,j}\frac{1}{|\Delta_i||\Delta_j|}\left(\int_{\Delta_i\times\Delta_j}X_{\underline{z}}\,\mathrm{d}\underline{z}\right)U(\Delta_i)U(\Delta_j)=A_1+2A_2+A_3.\tag{2.18}$$

We want now to check that $L^2-\lim_{n\to\infty}A_i$, $i=1,2,3$, exist.

It is easy to verify

$$L^2(\Omega) - \lim_{n\to\infty} A_3 = \int_{T^2} X_{r,s}(\nabla_0^1 u)_s(\nabla_0^1 u)_r \, dr \, ds. \tag{2.19}$$

Indeed, the assumptions on u ensure that the conclusion (2.2) of Lemma 2.1.1 holds true in the particular case

$$Y_{r,s} = (\nabla_0^1 u)_r(\nabla_0^1 u)_s.$$

We can also state

$$L^2 - \lim_{n\to\infty} A_2 = \delta\left(\int_T X_{\cdot,s} u_{\cdot}(\nabla_0^1 u)_s \, ds\right)$$

$$+ \int_{T^2} [X_{r,s} D_r(\nabla_0^1 u)_s + ((\nabla_0^1 X)_{(r,\cdot)})_s(\nabla_0^1 u)_s] u_r \, dr \, ds. \tag{2.20}$$

Indeed, Theorem 3.2 of Jolis and Sanz-Solé (1992) yields

$$A_2 = A_{21} + A_{22} + A_{23},$$

where

$$A_{21} = \sum_{i,j} \frac{1}{|\Delta_i||\Delta_j|} \delta\left\{\left(\int_{\Delta_i \times \Delta_j} X_{\underline{z}} \, d\underline{z}\right) 1_{\Delta_i}(r) u_r \left(\int_T 1_{\Delta_j}(s)(\nabla_0^1 u)_s \, ds\right)\right\},$$

$$A_{22} = \sum_{i,j} \frac{1}{|\Delta_i||\Delta_j|} \int_T 1_{\Delta_i}(r) u_r \left(\int_{\Delta_i \times \Delta_j} D_r X_{\underline{z}} \, d\underline{z}\right)\left(\int_T 1_{\Delta_j}(s)(\nabla_0^1 u)_s \, ds\right) dr,$$

$$A_{23} = \sum_{i,j} \frac{1}{|\Delta_i||\Delta_j|} \int_T 1_{\Delta_i}(r) u_r \left(\int_{\Delta_i \times \Delta_j} X_{\underline{z}} \, d\underline{z}\right)\left(\int_T 1_{\Delta_j}(s) D_r(\nabla_0^1 u)_s \, ds\right) dr.$$

Set $Y_{r,s} = u_r(\nabla_0^1 u)_s$. Lemma 2.1.1. (see (2.3)) yields

$$L^2 - \lim_{n\to\infty} A_{21} = \delta\left(\int_T X_{\cdot,s} u_{\cdot}(\nabla_0^1 u)_s \, ds\right).$$

Moreover

$$L^2 - \lim_{n\to\infty} A_{23} = \int_{T^2} X_{r,s} u_r D_r(\nabla_0^1 u)_s \, dr \, ds,$$

due to (2.2) with $Y_{r,s} = u_r D_r(\nabla_0^1 u)_s$. Finally,

$$L^2 - \lim_{n\to\infty} A_{22} = \int_{T^2} ((\nabla_0^1 X)_{(r,\cdot)})_s u_r(\nabla_0^1 u)_s \, dr \, ds.$$

Indeed, set $Y_{r,s} = u_r(\nabla_0^1 u)_s$. The assumptions of Lemma 2.1.2 are satisfied. Then, this last convergence is a particular case of (2.5). Therefore (2.20) is proved.

Finally, Lemma 2.1.3 yields the existence of $L^2 - \lim_{n\to\infty} A_1$, and that this limit equals the right-hand side of (2.7). Hence the integral $I_2^{s,U}$ exists.

Fubini's theorem for the Skorohod integrals with respect to the process U yields

$$\delta_2^U(X) = \delta^2(X_{r,s}u_r u_s) + \delta\left[\int_T X_{r,s}u_s D_s u_r \, ds\right] + 2\delta\left[\int_T X_{r,s}u_r(\nabla_0^1 u)_s \, ds\right]$$

$$+ \int_{T^2} X_{r,s}u_s D_s(\nabla_0^1 u)_r \, dr \, ds + \int_{T^2} X_{r,s}(\nabla_0^1 u)_s(\nabla_0^1 u)_r \, dr \, ds. \tag{2.21}$$

Then, a straightforward verification using the results (2.19)–(2.21) shows that

$$I_2^{s,U} = L^2 - \lim_{n\to\infty}(A_1 + 2A_2 + A_3)$$

coincides with the second term of (2.17). □

2.2. A Fubini theorem for the Stratonovich integral

Consider the process U defined by (1.6) with $v \equiv 0$. In this section we prove a Fubini theorem for the double Stratonovich integral with respect to U (see Theorem 2.2.2). The main ingredients of the proof are Proposition 2.1.5 (the Hu–Meyer formula) and Proposition 1.19 (the iteration of traces). We also need some details on the computations of kernels for traces of some products of the integrand X and the process u appearing in the integrator (see Lemma 2.2.1). For this reason we have to strengthen the assumption on X and u.

Here is the set of hypotheses to be considered.

(h) $u \in \mathbb{L}^{2,\infty} \cap \mathbb{L}_c^{1,2}$ and has a continuous version in L^2,
for a.e. r, $D_r u$ has a continuous version in L^2 and $D_r u \in \mathbb{L}_c^{1,2}$,
for a.e. (s,r), $D_{s,r}^2 u$ has a continuous version in L^2,
for any $i \in \{1,2,3,4\}$, $D^{(i)}u \in L^\infty(T\times\Omega)$ and $D^{(i)}(Du) \in L^\infty(T^2\times\Omega)$.

(H) X is a symmetric process satisfying

$$\operatorname*{ess\,sup}_{s,t,r,h}\{\|X_{t,h}\|_{L^4(\Omega)} + \|D_s X_{t,h}\|_{L^4(\Omega)} + D_{s,r}^2 X_{t,h}\|_{L^4(\Omega)}\} < \infty,$$

$X_{\cdot,h}$ has a continuous version in L^2, uniformly in h,
$D_r X_{\cdot,h}$ has a continuous version in L^2, uniformly in (r,h),
$D_{r,s}^2 X_{\cdot,h}$ has a continuous version in L^2, uniformly in (r,s,h),
for a.e. h, (r,h) and (r,s,h), respectively.

Lemma 2.2.1. *Consider two stochastic processes $X = \{X_{s,r}, (s,r)\in T^2\}$ and $u = \{u_s, s\in T\}$ satisfying the assumptions* (H) *and* (h), *respectively. Then, for a.e. $r\in T$, the processes $\{(\nabla_0^1(X_{\cdot,r}u_\cdot))_s, s\in T\}$, $\{(\nabla_0^1(X_{\cdot,r}u_\cdot u_r))_s, s\in T\}$ are well-defined. Furthermore. $\{(\nabla_1^0(Xuu))_r, r\in T\}$ and $\{\{\nabla_0^2(Xuu)\}_{s,r}, (s,r)\in T^2\}$ also exist. In addition,*

$$(\nabla_0^1(X_{\cdot,r}u_\cdot))_s = u_s(\nabla_0^1 X_{\cdot,r})_s + X_{s,r}(\nabla_0^1 u)_s, \tag{2.22}$$

$$(\nabla_0^1(X_{\cdot,r}u_\cdot u_r))_s = u_r(\nabla_0^1 X_{\cdot,r}u_\cdot))_s + X_{s,r}u_s D_s u_r, \tag{2.23}$$

$$(\nabla_1^0(Xuu))_r = u_r^2(\nabla_1^0 X)_r \tag{2.24}$$

and

$$(\nabla_0^2(Xuu))_{s,r} = (\nabla_0^2 X)_{s,r} u_s u_r + D_r X_{s,r} u_r D_s u_s + D_s X_{s,r} u_s D_r u_r + D_r X_{s,r} u_s D_s u_r + D_s X_{s,r} u_r D_r u_s + X_{s,r} (\nabla_0^1 (D_r u))_s u_r + X_{s,r} (\nabla_0^1 (D_s u))_r u_s + X_{s,r} D_r u_s D_s u_r + X_{s,r} D_s u_s D_r u_r. \tag{2.25}$$

Proof. We start with the proof of (2.22). It suffices to show that, for any $i \in \{1, 2, 3, 4\}$,

$$L^2(\Omega) - \lim_{\substack{t \to s \\ t R^i s}} D_s(X_{t,r} u_t) := (D^{(i)}(X_{\cdot,r} u_\cdot))_s$$

exists and

$$(D^{(i)}(X_{\cdot,r} u_\cdot))_s = (D^{(i)}(X_{\cdot,r})_s u_s + X_{s,r}(D^{(i)} u)_s. \tag{2.26}$$

Let

$$\alpha_1 = E(|u_t D_s X_{t,r} - u_s (D^{(i)} X_{\cdot,r})_s|^2)$$

and

$$\alpha_2 = E(|X_{t,r} D_s u_t - X_{s,r} (D^{(i)} u)_s|^2).$$

We want to prove

$$\lim_{\substack{t \to s \\ t R^i s}} \alpha_1 = 0, \qquad \lim_{\substack{t \to s \\ t R^i s}} \alpha_2 = 0. \tag{2.27}$$

Since

$$E|D_s(X_{t,r} u_t) - [u_s (D^{(i)} X_{\cdot,r})_s + X_{s,r} (D^{(i)} u)_s]|^2 \leqslant 2(\alpha_1 + \alpha_2),$$

(2.27) establishes (2.22).

We have

$$\alpha_1 \leqslant 2(\alpha_{11} + \alpha_{12}) \tag{2.28}$$

with

$$\alpha_{11} = E(|u_t - u_s|^2 |D_s X_{t,r}|^2),$$

$$\alpha_{12} = E(|u_s|^2 |D_s X_{t,r} - (D^{(i)} X_{\cdot,r})_s|^2).$$

Since u is bounded, continuous in L^2 and $\operatorname{ess\,sup}_{s,t,r} \|D_s X_{t,r}\|_{L^4(\Omega)} < \infty$, Schwarz's inequality yields

$$\lim_{\substack{t \to s \\ t R^i s}} \alpha_{11} = 0.$$

Analogously

$$\alpha_{12} \leqslant C E(|D_s X_{t,r} - (D^{(i)} X_{\cdot,r})_s|^2).$$

Consequently

$$\lim_{\substack{t \to s \\ t R^i s}} \alpha_{12} = 0$$

and, therefore, $\lim_{\substack{t \to s \\ t R^i s}} \alpha_1 = 0$, by (2.28). The proof of $\lim_{\substack{t \to s \\ t R^i s}} \alpha_2 = 0$ is analogous.

Let us now prove the statement (2.23). That means, for any $i \in \{1,2,3,4\}$,

$$L^2(\Omega) - \lim_{\substack{t \to s \\ t R^i s}} D_s(X_{t,r} u_t u_r) := (D^{(i)}(X_{\cdot,r} u_\cdot u_r))_s \tag{2.29}$$

exists and

$$(D^{(i)}(X_{\cdot,r} u_\cdot u_r))_s = u_r (D^{(i)}(X_{\cdot,r} u_\cdot))_s + X_{s,r} u_s D_s u_r. \tag{2.30}$$

Let

$$\beta_1 = E(|u_r[D_s(X_{t,r} u_t) - (D^{(i)}(X_{\cdot,r} u_\cdot))_s]|^2)$$

and

$$\beta_2 = E(|D_s u_r [X_{t,r} u_t - X_{s,r} u_s]|^2).$$

Since the process Xu, defined by $(Xu)_{r,s} = X_{r,s} u_s$, belongs to $\mathbb{L}^{1,2}_{2,c(0,1)}$ and u is bounded, it holds $\lim_{\substack{t \to s \\ t R^i s}} \beta_1 = 0$. Furthermore, since $D_s u_r$ is bounded the continuity properties of X and u yield $\lim_{\substack{t \to s \\ t R^i s}} \beta_2 = 0$. This shows the validity of (2.30). The proof of (2.24) is analogous.

To prove (2.25) we develop the derivative $D_{r,s}(X_{t,h} u_t u_h)$ and we show the convergence of each term in this development to the right-hand side of (2.25), as $t \to r$, $h \to s$, $t R^i r$, $h R^j s$, $i, j \in \{1,2,3,4\}$, using repeatedly the same ideas as before. □

We can now state Fubini's theorem.

Theorem 2.2.2. *Let $X = \{X_z, z \in T^2\}$ be a symmetric process and $U_z = \int_{R_z} u_r \circ \mathrm{d}W_r$. Suppose that the process X and u satisfy conditions* (H) *and* (h), *respectively. Then, $X \in Dom\, I_2^{s,U}$ and*

$$\int_{T^2} X_{s,r} \circ \mathrm{d}U_s \circ \mathrm{d}U_r = \int_T \left(\int_T X_{s,r} \circ \mathrm{d}U_s \right) \circ \mathrm{d}U_r.$$

Remark 2.2.3. The hypotheses of the previous theorem imply

$$\delta(X_{\cdot,r} u_\cdot) \in \mathbb{L}_c^{1,2}, \qquad \delta(X_{\cdot,r} u_\cdot u_r) \in \mathbb{L}_c^{1,2}.$$

Proof of Theorem 2.2.2. In a first step we will show that the iterated integral $I^{s,U}(I^{s,U}(X))$ exists and we will give its value (see (2.35)). In a second step we will check the equality between the iterated and the double integral using Proposition 2.1.5.

Step 1: The assumptions on X and u yield $X_{\cdot,r} \in \mathbb{L}_c^{1,2}$ and $X_{\cdot,r}u_{\cdot} \in \mathbb{L}_c^{1,2}$ for a.e. $r \in T$. Hence, by Proposition 1.9 (see (1.12) and (1.11)) $X_{\cdot,r} \in Dom\, I^{s,U}$ and

$$I^{s,U}(X_{\cdot,r}) = \delta(X_{\cdot,r}u) + T_{0,1}(X_{\cdot,r}u). \tag{2.31}$$

We want to prove that $I^{s,U}(X_{\cdot,r})$ belongs to $\mathbb{L}_c^{1,2}$ as well as $I^{s,U}(X_{\cdot,r})u_r$.

The equality (2.22) ensures

$$\begin{aligned} T_{0,1}(X_{\cdot,r}u) &= \int_T (\nabla_0^1 X_{\cdot,r})_s u_s \, ds + \int_T X_{s,r}(\nabla_0^1 u)_s \, ds \\ &= \int_T D_s X_{s,r} u_s \, ds + \int_T X_{s,r} D_s u_s \, ds. \end{aligned} \tag{2.32}$$

Therefore $T_{0,1}(X_{\cdot,r}u) \in \mathbb{L}_c^{1,2}$. Hence, taking into account Remark 2.2.3 and (2.31), $I^{s,U}(X_{\cdot,r}) \in \mathbb{L}_c^{1,2}$. The identity (2.31) yields

$$I^{s,U}(X_{\cdot,r})u_r = \delta(X_{\cdot,r}u_{\cdot}u_r) + \int_T X_{s,r}u_s D_s u_r \, ds + u_r T_{0,1}(X_{\cdot,r}u). \tag{2.33}$$

We next prove

$$\int_T X_{s,r}u_s D_s u_r \, ds + u_r T_{0,1}(X_{\cdot,r}u) = T_{0,1}(Xuu), \tag{2.34}$$

where Xuu is defined in (2.8). Notice that $(T_{0,1}(Xuu))_r = T_{0,1}(X_{\cdot,r}u_{\cdot}u_r)$. Indeed, the hypotheses on X and u ensure that $Xuu \in \mathbb{L}_{2,c(0,1),u}^{1,2}$ and $Xu \in \mathbb{L}_{2,c(0,1),u}^{1,2}$. Consequently, (1.18) and (2.23) yield the existence of $T_{0,1}(Xuu)$ and

$$\begin{aligned} (T_{0,1}(Xuu))_r &= \int_T (\nabla_0^1(X_{\cdot,r}u_{\cdot}u_r))_s \, ds = \int_T u_r(\nabla_0^1(X_{\cdot,r}u_{\cdot}))_s \, ds \\ &\quad + \int_T X_{s,r}u_s D_s u_r \, ds. \\ &= u_r T_{0,1}(X_{\cdot,r}u_{\cdot}) + \int_T X_{s,r}u_s D_s u_r \, ds. \end{aligned}$$

It is also clear that $T_{0,1}(Xuu) \in \mathbb{L}_c^{1,2}$. Then, substituting (2.34) into (2.33) we obtain

$$I^{s,U}(X_{\cdot,r})u_r = \delta(X_{\cdot,r}u_{\cdot}u_r) + T_{0,1}(X_{\cdot,r}u_{\cdot}u_r).$$

Therefore, $I^{s,U}(X_{\cdot,r})u_r \in \mathbb{L}_c^{1,2}$ (see Remark 2.2.3). By Proposition 1.9 we conclude that $I^{s,U}(X_{\cdot,r}) \in Dom\, I^{s,U}$ and, by (1.12),

$$I^{s,U}(I^{s,U}(X_{s,r})) = \delta(I^{s,U}(X_{\cdot,r})u_r) + T_{0,1}(I^{s,U}(X_{\cdot,r})u_r). \tag{2.35}$$

Step 2: The right-hand side of (2.35) can be developed using (2.31) as follows:

$$\begin{aligned} I^{s,U}(I^{s,U}(X_{s,r})) &= \delta^2(X_{s,r}u_s u_r) + \delta\left(\int_T X_{s,r}u_s D_s u_r \, ds\right) + \delta(T_{0,1}(X_{\cdot,r}u_{\cdot})u_r) \\ &\quad + T_{0,1}(\delta(X_{s,r}u_s u_r)) + T_{0,1}(T_{0,1}(Xuu)). \end{aligned} \tag{2.36}$$

Let

$$A_1 = \delta(T_{0,1}(X_{\cdot,r}u.)u_r), \quad A_2 = T_{0,1}(T_{0,1}(X_{\cdot,r}u.u_r)), \quad A_3 = T_{0,1}(\delta(X_{\cdot,r}u.u_r)).$$

The identity (2.32) yields

$$A_1 = \delta\left(u_r \int_T [D_s X_{s,r} u_s + X_{s,r} D_s u_s] \, \mathrm{d}s\right). \tag{2.37}$$

The process $(Xuu)_{r,s} = X_{r,s} u_r u_s$ satisfies the assumptions of Proposition 1.19. Hence,

$$A_2 = T_{0,1}(T_{0,1}(Xuu)) = T_{0,2}(Xuu).$$

Moreover, (1.18), (2.25) and the continuity properties of $D^2_{s,r}u$ yield

$$\begin{aligned} A_2 = &\int_{T^2} (\nabla_0^2 X)_{s,r} u_s u_r \, \mathrm{d}s \, \mathrm{d}r + 2\int_{T^2} D_r X_{s,r} u_r D_s u_s \, \mathrm{d}r \, \mathrm{d}s + 2\int_{T^2} D_r X_{s,r} u_s D_s u_r \, \mathrm{d}s \, \mathrm{d}r \\ &+ 2\int_{T^2} X_{s,r} D^2_{s,r} u_s u_r \, \mathrm{d}s \, \mathrm{d}r + \int_{T^2} X_{s,r} D_r u_s D_s u_r \, \mathrm{d}s \, \mathrm{d}r \\ &+ \int_{T^2} X_{s,r} D_s u_s D_r u_r \, \mathrm{d}s \, \mathrm{d}r. \end{aligned} \tag{2.38}$$

The process Xuu belongs to $\mathbb{L}^{1,2}_{2,c(0,1)} \cap \mathbb{L}^{1,2}_{2,c(1,0)}$. Then, Proposition 1.21, (2.34), (2.32) and (2.24) yield

$$\begin{aligned} A_3 &= \delta(T_{0,1}(Xuu)) + T_{1,0}(Xuu) \\ &= \delta\left\{\int_T [D_s X_{s,\cdot} u_s u_\cdot + X_{s,\cdot} D_s u_s u_\cdot + X_{s,\cdot} u_s D_s u_\cdot] \, \mathrm{d}s\right\} + \int_T X_{r,r} u_r^2 \, \mathrm{d}r. \end{aligned} \tag{2.39}$$

Then, replacing in (2.36) the expressions of A_1 to A_3 given in (2.37) to (2.39) we obtain

$$\begin{aligned} I^{s,U}(I^{s,U}(X_{s,r})) = &\delta^2(X_{s,r} u_s u_r) + 2\delta\left(\int_T X_{s,r} u_s D_s u_r \, \mathrm{d}s\right) \\ &+ 2\delta\left(\int_T D_s X_{s,r} u_s u_r \, \mathrm{d}s\right) + 2\delta\left(\int_T X_{s,r} D_s u_s u_r \, \mathrm{d}s\right) \\ &+ \int_T X_{r,r} u_r^2 \, \mathrm{d}r + \int_{T^2} (\nabla_0^2 X)_{s,r} u_s u_r \, \mathrm{d}s \, \mathrm{d}r \\ &+ 2\int_{T^2} D_s X_{s,r} u_s D_r u_r \, \mathrm{d}s \, \mathrm{d}r \\ &+ 2\int_{T^2} D_s X_{s,r} u_r D_r u_s \, \mathrm{d}s \, \mathrm{d}r + 2\int_{T^2} X_{s,r} D^2_{s,u} u_s u_r \, \mathrm{d}s \, \mathrm{d}r \\ &\int_{T^2} X_{s,r} D_r u_r D_s u_s \, \mathrm{d}s \, \mathrm{d}r + \int_{T^2} X_{s,r} D_s u_r D_r u_s \, \mathrm{d}s \, \mathrm{d}r. \end{aligned} \tag{2.40}$$

In order to finish the proof we notice that the hypotheses of Lemma 2.1.4 and Proposition 2.1.5 are satisfied; hence, $X \in Dom\, I_2^{s,U}$, and using (2.17) and (2.21) we check that the right-hand side of (2.40) coincides with $I_2^{s,U}(X)$. The theorem is now completely proved. □

2.3. The main theorem

This section is devoted to proving a Green type formula on rectangles in the Stratonovich sense. First we should introduce a kind of stochastic line integral (see Cairoli and Walsh (1975) for the nonanticipating case). Given $z_0 \in T$, we denote by H_{z_0} (resp. B_{z_0}) the horizontal (resp. vertical) line segment connecting z_0 and the y-axis (resp. x-axis). Let $z_0 = (s_0, t_0)$, by definition, $X^{H_{z_0}}$ and $X^{B_{z_0}}$ are the processes indexed by $z = (s,t) \in T$ given by

$$X_{s,t}^{H_{z_0}} = X_{s,t_0} \mathbf{1}_{R_{s_0,t_0}}(s,t) \quad \text{and} \quad X_{s,t}^{B_{z_0}} = X_{s_0,t} \mathbf{1}_{R_{s_0,t_0}}(s,t),$$

respectively.

In the next definition $X = \{X_z, z \in T\}$ is a measurable process and $U = (u^s, v)$ are the stochastic processes given by (1.6).

Definition 2.3.1. The process X is said to be *Stratonovich integrable along* H_{z_0}(resp. B_{z_0}) *with respect to* U if $X^{H_{z_0}} \in Dom\, I^{s,U}$ (resp. $X^{B_{z_0}} \in Dom\, I^{s,U}$). In this case the integral is defined as $I^{s,U}(X^{H_{z_0}})$ (resp. $I^{s,U}(X^{B_{z_0}})$) and denoted by $\int_{H_{z_0}} X \circ \partial_1 U$ (resp $\int_{B_{z_0}} X \circ \partial_2 U$).

We denote by $Dom\, I_{H_{z_0}}^{s,U}$ (resp. $Dom\, I_{B_{z_0}}^{s,U}$) the set of processes X for which $\int_{H_{z_0}} X \circ \partial_1 U$ (resp. $\int_{B_{z_0}} X \circ \partial_2 U$) exists.

By additivity we extend these definitions to a compact curve which is a finite union of horizontal and vertical line segments. In particular, if $(a_1, b_1) < (a_2, b_2)$ are two ordered points in T, A denotes the rectangle $[(a_1,b_1),(a_2,b_2)]$ and ∂A the boundary of A oriented in the clockwise direction, we set

$$\int_{\partial A} X \circ \partial_1 U := \int_{H_{a_2,b_2}} X \circ \partial_1 U - \int_{H_{a_2,b_1}} X \circ \partial_1 U - \int_{H_{a_1,b_2}} X \circ \partial_1 U + \int_{H_{a_1,b_1}} X \circ \partial_1 U. \tag{2.41}$$

Given a measurable process $X = \{X_z, \ z \in T\}$, we define the process $Y = \{Y_{z,z'}, (z,z') \in T^2\}$ by

$$Y_{z,z'} = X_{z \vee z'} \mathbf{1}_{z \perp z'}.$$

We now introduce a special case of double anticipating integral.

Definition 2.3.2. The process X is said to be *J-integrable, in the Stratonovich sense, with respect to* U if $Y \in Dom\, I_2^{s,U}$. In this case we define the J-Stratonovich integral of X with respect to U by

$$I_J^{s,U}(X) = I_2^{s,U}(Y).$$

This integral will also be denoted by $\int_T X \circ \mathrm{d}J_U$. The set of processes X for which this integral exists is denoted by $Dom\, I_J^{s,U}$.

The following definition precises the meaning of stochastic derivative used along this section.

Definition 2.3.3. Consider two measurable processes $f = \{f_z,\ z \in T\}$ and $g = \{g_z,\ z \in T\}$ such that

(i) for any $z \in T$, $f \in Dom\, I_{B_z}^{s,U}$,

(ii) for a.e. $s \in [0,1]$, $\int_0^1 |g_{s,\eta}|\,\mathrm{d}\eta < \infty$, a.s.

Let $X = \{X_z,\ z \in T\}$ be the process defined by

$$X_{s,t} = X_{s,0} + \int_{B_{s,t}} f \circ \partial_2 U + \int_0^t g_{s,\eta}\,\mathrm{d}\eta.$$

Then, X is said to possess 2-*Stratonovich stochastic partial derivatives* f and g in T with respect to U.

The corresponding definitions of the 1-Stratonovich stochastic partial derivatives with respect to U are obtained by replacing B_z by H_z, ∂_2 by ∂_1 and condition (ii) by:

(ii′) For a.e. $t \in [0,1]$, $\int_0^1 |g_{\eta,t}|\,\mathrm{d}\eta < \infty$, a.s.

As in the preceding section, we will assume that $U_z = \int_{R_z} u_r \circ \mathrm{d}W_r$ for the sake of simplicity.

Theorem 2.3.4. *Let $X = \{X_z,\ z \in T\}$ be a measurable process, null on the axes, possessing* 2-*Stratonovich* stochastic partial derivatives, *f and g, in T with respect to U. Consider the process $\varphi = \{\varphi_{z,z'},\ (z,z') \in T^2\}$ defined by*

$$\varphi_{z,z'} = f_{z \vee z'} \mathbf{1}_{z \perp z'},$$

and its symmetrization, $\tilde{\varphi}$.

We assume that the process $X\mathbf{1}_A$ belongs to $\mathbb{L}_c^{1,2}$, the process u satisfies (h) *and $\tilde{\varphi} \cdot \mathbf{1}_{A \times A}$ satisfies* (H). *Furthermore, suppose that $X^{\partial A}$ belongs to $\mathbb{L}_c^{1,2}$. Then, $X \in Dom\, I_{\partial A}^{s,U}$ and*

$$\int_{\partial A} X \circ \partial_1 U = \int_A X \circ \mathrm{d}U + \int_A f \circ \mathrm{d}J_U + \int_A \left(\int_t^{b_2} g_{s,r}\,\mathrm{d}r \right) \circ \mathrm{d}U_{s,t}, \tag{2.42}$$

where $\int_{\partial A} X \circ \partial_1 U$ is given by (2.41) *and*

$$\int_A X \circ \mathrm{d}U = I^{s,U}(X\mathbf{1}_A), \qquad \int_A f \circ \mathrm{d}J_U = I_2^{s,U}(\tilde{\varphi}\,\mathbf{1}_{A \times A}).$$

Proof. We will assume $A = T$ and X null on the axis, since this does not represent a restriction. Proposition 1.9 yields $X \in Dom\, I_{H_{1,1}}^{s,U}$. We want to check that

$$\int_{H_{1,1}} X \circ \partial_1 U = \int_T X \circ \mathrm{d}U + \int_T f \circ \mathrm{d}J_U + \int_T \left(\int_t^1 g_{s,r}\,\mathrm{d}r \right) \circ \mathrm{d}U_{s,t} \tag{2.43}$$

with

$$X_{s,t} = \int_{B_{s,t}} f \circ \partial_2 U + \int_0^t g_{s,r}\, \mathrm{d}r.$$

By definition,

$$\int_{H_{1,1}} X \circ \partial_1 U = \int_T X_z^{H_{1,1}} \circ \mathrm{d}U.$$

We decompose the process $X_z^{H_{1,1}}$, $z = (s,t)$ as follows:

$$\begin{aligned} X_z^{H_{1,1}} &= X_{s,1} = \int_{B_{s,1}} f \circ \partial_2 U + \int_0^1 g_{s,r}\, \mathrm{d}r \\ &= \int_{R_s} f_{s,y} \circ \mathrm{d}U_{x,y} + \int_0^t g_{s,r}\, \mathrm{d}r + \int_{R_{s,1} \setminus R_s} f_{s,y} \circ \mathrm{d}U_{x,y} + \int_t^1 g_{s,r}\, \mathrm{d}r \\ &= X_z + \int_T f_{(x,y) \vee (s,t)}\, \mathbf{1}_{[0,s]}(x)\, \mathbf{1}_{[t,1]}(y) \circ \mathrm{d}U_{x,y} + \int_0^1 g_{s,r}\, \mathbf{1}_{R_{1,r}}(z)\, dr \\ &= X_z + \int_T \varphi_{z',z} \circ \mathrm{d}U_{z'} + \int_t^1 g_{s,r}\, \mathrm{d}r. \end{aligned}$$

Hence,

$$\int_{H_{1,1}} X \circ \partial_1 U = \int_T X \circ \mathrm{d}U + \int_T \left(\int_T \tilde{\varphi}_{z,z'} \circ \mathrm{d}U_{z'} \right) \circ \mathrm{d}U_z + \int_T \left(\int_t^1 g_{s,r}\, \mathrm{d}r \right) \circ \mathrm{d}U_z.$$

Moreover, the process $\tilde{\varphi}$ and u satisfy the assumptions of Theorem 2.2.2, hence

$$\int_T \left(\int_T \tilde{\varphi}_{z,z'} \circ \mathrm{d}U_{z'} \right) \circ \mathrm{d}U_z = I_2^{s,U}(\tilde{\varphi}) = \int_T f \circ \mathrm{d}J_U$$

and therefore formula (2.43) is completely proved. □

Remark 2.3.5. As can be easily checked by the reader the Green formula (2.42) does not reduce to Solé and Utzet's formula (Theorem 4.25 of Solé and Utzet, 1991) when $u \equiv 1$ and $v \equiv 0$. The reason is that our definitions of the J-integral do not coincide with theirs. However our choice does not lead to extra terms in comparison with the Skorohod formulation and, therefore, seems to be appropriate.

Remark 2.3.6. Assume $v \neq 0$, $v \in \mathbb{L}^{1,\infty}$. Fubini's theorem and (2.42) can be easily generalized to this situation.

2.4. Application

In this section we derive an Itô formula as a simple application of the Green formula proved in the preceding section.

We will use a regularization of the Wiener process obtained by convolution (see Thieullen, 1991). More precisely, let $\rho = \{\rho_n, n \geqslant 1\}$ be a Dirac sequence on $[-1,1]^2$, ρ_n positive, bounded, symmetric with respect to the axes, $\rho_n(u) = 0\ \forall u \notin [-1/n, 1/n]^2$ and $\int_{[-1/n,1/n]^2} \rho_n(u)\,du = 1$, for any $n \geqslant 1$. Set $W_z^n = \int_T \rho_n(z-r)\,dW_r$. Propositions 4.6 and 4.4 of Thieullen (1991), together with Theorem 4.7 of Solé and Utzet (1991), yield that if $X \in \mathbb{L}_c^{1,2}$, then the sequence $\{\int_T X_z W_z^n dz,\ n \geqslant 1\}$ converges in $L^2(\Omega)$ to $\int_T X_z \circ dW_z$. First we present a "line" Itô formula.

Proposition 2.4.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a $\mathscr{C}^1$ function and $U_z = \int_{R_z} u_r \circ dW_r$, $z \in T$. Assume*

(i) *$u \in \mathbb{L}^{1,\infty} \cap \mathbb{L}_c^{1,2}$, $D^{(i)}u \in L^\infty(T \times \Omega)$, $\forall i = 1,2,3,4$, and u has a continuous version in $L^2(\Omega)$,*

(ii) *the process U has a continuous version,*

(iii) *there exists $z \in T$ such that the processes $(\varphi'(U))^{H_z}$ and $(\varphi'(U))^{H_z}u$ belong to $\mathbb{L}_c^{1,2}$.*

Then

$$\varphi(U_z) = \varphi(0) + \int_{H_z} \varphi'(U) \circ \partial_1 U, \quad \text{a.s.} \tag{2.44}$$

Proof. Set $z = (s,t)$ *and* $U_z^n = \int_{R_z} u_r W_r^n dr$. The deterministic change of variable formula yields

$$\varphi(U_z^n) = \varphi(0) + \int_0^s \varphi'(U_{x,t}^n)\left(\int_0^t u_{x,y} W_{x,y}^n dy\right) dx.$$

Then (2.44) follows from the following facts,

(a) $\varphi(U_z^n) \xrightarrow[n\to\infty]{} \varphi(U_z)$

and

(b) $\displaystyle\int_0^s \varphi'(U_{x,t}^n)\left(\int_0^t u_{x,y} W_{x,y}^n dy\right) dx \xrightarrow[n\to\infty]{} \int_{H_z} \varphi'(U) \circ \partial_1 U,$

in probability.

The first one is obvious, since $\lim_{n\to\infty} U_z^n = U_z$ in $L^2(\Omega)$. Analogously, $\varphi'(U_z^n) \xrightarrow[n\to\infty]{} \varphi'(U_z)$ in probability. Let us now prove (b). Using a localization argument we can assume that φ' is bounded. Let

$$b_1^n = E\left(\left|\int_0^s [\varphi'(U_{x,t}^n) - \varphi'(U_{x,t})]\left(\int_0^t u_{x,y} W_{x,y}^n dy\right) dx\right|\right),$$

$$b_2^n = E\left(\left|\int_{R_z} \varphi'(U_{x,t}) u_{x,y} W_{x,y}^n dx\,dy - \int_{R_z} \varphi'(U_{x,t}) u_{x,y} \circ dW_{x,y}\right|\right).$$

Using Proposition 1.9 (1.12), we obtain

$$E\left(\left|\int_0^s \varphi'(U_{x,t}^n)\left(\int_0^1 u_{x,y} W_{x,y}^n dy\right) dx - \int_{H_z} \varphi'(U) \circ \partial_1 U\right|\right) \leqslant C(b_1^n + b_2^n).$$

The same arguments leading to the convergence $L^2-\lim_{n\to\infty} U_z^n = U_z$ yield $\lim_{n\to\infty} b_2^n = 0$. Finally, Schwarz's inequality implies

$$b_1^n \leqslant E\left[\int_0^s |(\varphi'(U_{x,t}^n) - \varphi'(U_{x,t}))\left(\int_0^t u_{x,y} W_{x,y}^n \mathrm{d}y\right)| \, \mathrm{d}x\right]$$

$$\leqslant \alpha_n \beta_n,$$

where

$$\alpha_n = \left(E\left[\int_0^s |(\varphi'(U_{x,t}^n) - \varphi'(U_{x,t}))|^2 \, \mathrm{d}x\right]\right)^{1/2},$$

$$\beta_n = \left(E\left[\int_0^s \left|\int_0^t u_{x,y} W_{x,y}^n \mathrm{d}y\right|^2 \mathrm{d}x\right]\right)^{1/2}.$$

The sequence $\{\alpha_n,\ n \geqslant 1\}$ converges to zero as n tends to infinity, by dominated convergence. Moreover, $\sup_n \beta_n < \infty$. Indeed, since u is bounded, Schwarz's inequality yields

$$\sup_n \beta_n^2 \leqslant \sup_n C \int_{R_{s,t}} E|W_{x,y}^n|^2 \, \mathrm{d}x \, \mathrm{d}y \leqslant C \sup_n \sup_{z\in T} E|W_z^n|^2 < \infty.$$

This finishes the proof of the proposition. □

Remarks.

(2.4.2) Hypothesis (ii) of Proposition 2.4.1 is used in order to justify the localization procedure. Conditions for the continuity of U can be derived from Proposition 4.8 of Jolis and Sanz-Solé (1990).

(2.4.3) Assume, instead of assumption (iii), the analogue

(iii′) There exists $z \in T$ such that the processes $(\varphi'(U))^{B_z}$ and $(\varphi'(U))^{B_z} u$ belong to $\mathbb{L}_c^{1,2}$. Then we obtain

$$\varphi(U_z) = \varphi(0) + \int_{B_z} \varphi'(U) \circ \partial_2 U, \quad \text{a.s.} \tag{2.45}$$

We can now, using the Green–Stratonovich formula, derive the Itô–Stratonovich formula stated in Thieullen (1991, Theorem 5.4), for $v = 0$.

Theorem 2.4.4. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function of class $\mathscr{C}^2$ and let $U_z = \int_{R_z} u_r \circ \mathrm{d}W_r$, $z \in T$. Suppose:*

(i) *the process u verifies* (h),

(ii) *the process U has a continuous version,*

(iii) $\varphi'(U)\, \mathbf{1}_{R_z} \in \mathbb{L}_c^{1,2}$,

(iv) *for any $z \in T$, the processes $(\varphi'(U))^{H_z}$ and $(\varphi'(U))^{H_z} u$ belong to $\mathbb{L}_c^{1,2}$,*

(v) *for any $z \in T$, the processes $(\varphi''(U))^{B_z}$ and $(\varphi''(U))^{B_z} u$ belong to $\mathbb{L}_c^{1,2}$,*

(vi) *if $h_{r,r'} := \varphi''(U_{r\vee r'})\, \mathbf{1}_{r \perp r'}$, $(r,r') \in T^2$, the process $\tilde{h}\, \mathbf{1}_{R_z \times R_z}$ satisfies the set of hypotheses* (H), *where $\tilde{h}$ denotes the symmetrization of h.*

Then,

$$\varphi(U_z) = \varphi(0) + \int_{R_z} \varphi'(U) \circ \mathrm{d}U + \int_{R_z} \varphi''(U) \circ \mathrm{d}J_U, \quad \text{a.s.} \tag{2.46}$$

Proof. Since the hypotheses of Proposition 2.4.1 are satisfied the process $\varphi(U_z)$ verifies (2.44). Moreover, (2.45) applied to the process U and the function φ' yields

$$\varphi'(U_z) = \varphi'(0) + \int_{B_z} \varphi''(U) \circ \partial_2 U,$$

a.s. for any $z \in T$. Hence, the process $X = \{\varphi'(U_z),\ z \in T\}$ has 2-Stratonovich stochastic partial derivatives with respect to U given by $f = \varphi''(U)$ and $g \equiv 0$.

Set $A = R_z$; the process X satisfies the hypotheses of Theorem 2.3.4. Consequently, $\varphi'(U) \in Dom\, I_{H_z}^{s,U}$ and

$$\int_{H_z} \varphi'(U) \circ \partial_1 U = \int_{R_z} \varphi'(U) \circ \mathrm{d}U + \int_{R_z} \varphi''(U) \circ \mathrm{d}J_U, \quad \text{a.s.}$$

Finally, replacing this expression in (2.44) we obtain

$$\varphi(U_z) = \varphi(0) + \int_{R_z} \varphi'(U) \circ \mathrm{d}U + \int_{R_z} \varphi''(U) \circ \mathrm{d}J_U, \quad \text{a.s.} \qquad \square$$

Final remarks.

(2.4.5) We can also introduce the Skorohod analogue of line, J-integrals and stochastic partial derivatives (see Definitions 2.3.1–2.3.3), and obtain the Skorohod version of the Green formula in a similar way. More precisely, let $X = \{X_z, z \in T\}$ be a version of a measurable process with 2-Skorohod stochastic partial dervatives ρ and g in T with respect to $V_z = \int_{R_z} u_r \, \mathrm{d}W_r$ and ψ be the process defined by $\psi_{z,z'} = \rho_{z \vee z'} \mathbf{1}_{z \perp z'}$; denote by $\tilde{\psi}$ the symmetrization of ψ. Then,

$$\int_{\partial A} X \,\partial_1 V = \int_A X \,\mathrm{d}V + \int_A \rho \,\mathrm{d}J_V + \int_A \left(\int_t^{b_2} g_{s,r} \,\mathrm{d}r \right) \mathrm{d}V_{s,t}, \tag{2.47}$$

where

$$\int_{\partial A} X \,\partial_1 V = \int_{H_{a_2,b_2}} X \,\partial_1 V - \int_{H_{a_2,b_1}} X \,\partial_1 V - \int_{H_{a_1,b_2}} X \,\partial_1 V - \int_{H_{a_1,b_1}} X \,\partial_1 V,$$

$$\int_A X \,\mathrm{d}V = \delta^V(X.\mathbf{1}_A), \qquad \int_A \rho \,\mathrm{d}J_V = \delta_2^V(\tilde{\psi}.\mathbf{1}_{A \times A}).$$

This is a generalization of Theorem 3.16 in Solé and Utzet (1991).

(2.4.6) From (2.47) we can derive, as we did in Theorem 2.4.4, an Itô–Skorohod formula. We obtain an alternative rather compact form for this change of variables formula (see Jolis and Sanz-Solé (1990) and Thieullen (1991) for related results).

Theorem 2.4.7 (Itô–Skorohod). *Let* $\varphi:\mathbb{R}^2 \to \mathbb{R}$ *be a continuous function of class* $\mathscr{C}^2$, *and* $V_z = \int_{R_z} u_r \,\mathrm{d}W_r$. *Suppose that:*

(i) $u \in \mathbb{L}_c^{1,2} \cap \mathbb{L}^{1,\infty}$, $D^{(i)}u \in L^\infty(T\times\Omega)$ $\forall i = 1,2,3,4$, *and* u *has a continuous version in* $L^2(\Omega)$,

(ii) *the process* V *has a continuous version,*

(iii) $\varphi'(V) \in L^2(T\times\Omega)$, $\varphi'(V)u\mathbf{1}_{R_z} \in Dom\,\delta$,

(iv) $(\varphi'(V))^{H_z} \in \mathbb{L}_c^{1,2}$ *and* $(\varphi'(V))^{H_z}u \in \mathbb{L}_c^{1,2}$, *for any* $z \in T$,

(v) $(\varphi''(V))^{B_z} \in \mathbb{L}_c^{1,2}$ *and* $(\varphi''(V))^{B_z}u \in \mathbb{L}_c^{1,2}$, *for any* $z \in T$,

(vi) *if* $\tilde{h}$ *is the symmetrization of* $h_{r,r'} =: \varphi''(V_{r\vee r'})\,\mathbf{1}_{r\perp r'}$,

$$\tilde{h}\mathbf{1}_{R_z\times R_z} \in L^2(T\times\Omega), \qquad \tilde{h}\mathbf{1}_{R_z\times R_z}\, u \in Dom\,\delta,$$

$$\delta^V(\tilde{h}\mathbf{1}_{R_z\times R_z}) \in L^2(T\times\Omega), \qquad \delta^V(\tilde{h}\mathbf{1}_{R_z\times R_z})\, u \in Dom\,\delta.$$

Then,

$$\begin{aligned}\varphi(V_z) = \varphi(0) &+ \int_{R_z} \varphi'(V)\,\mathrm{d}V + \int_{R_z} \varphi''(V)\,\mathrm{d}J_V \\ &+ \int_{R_z}\left(\int_{R_z} \mathbf{1}_{r\perp r'}\cdot(\nabla_0^1 \varphi''(V_{\cdot\vee r'}))_r u_r\,\mathrm{d}r\right)\mathrm{d}V_{r'} \\ &+ \int_{R_z} (\nabla_0^1(\varphi'(V_\cdot)^{(H_z)_1}))_r u_r\,\mathrm{d}r \quad \text{a.s.}\end{aligned}$$

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